Finding Hyperexponential Solutions of Linear ODEs by Numerical Evaluation

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Linear ODEs

Consider a linear differential operator

$$P = p_r D^r + p_{r-1} D^{r-1} + \ldots + p_1 D + p_0$$

where

- $D = \dfrac{d}{dx}$
- $p_k \in C[x]$
- C is an algebraically closed, computable subfield of $\mathbb{C}$

We want to find the hyperexponential solutions of $Py = 0$.

Hyperexponential solutions

A solution y of the equation $Py = 0$ is called **hyperexponential** if

$$\frac{Dy}{y} \in C(x).$$

Equivalently, y is hyperexponential iff

$$y = \exp(\textstyle\int v), \quad v \in C(x).$$

Examples: $\dfrac{5x+1}{3x+5}, \quad \sqrt{x+1}, \quad (x+1)^{\sqrt{2}} \exp\left(\dfrac{x^9}{x-1}\right)$

Exponential and rational parts

Finding a hyperexponential solution h is easy if we know its **exponential part** ($\approx h$ up to multiplication by a rational function):

$$h = \exp\left(\frac{1}{1-x}\right)\frac{(x+1)^2}{(x+2)(x+3)}$$

Make the ansatz $h = \exp\left(\frac{1}{1-x}\right)u$ and look for rational solutions u.

Finding rational solutions is **easy**. The difficulty is to find the possible exponential parts.

Local solutions

For each $z \in C \cup \{\infty\}$ the equation $Py = 0$ has a basis of r linearly independent (formal) **local solutions**.

The local solutions are generalized power series in

$$\tilde{x} = \begin{cases} x - z & \text{if } z \neq \infty \\ 1/x & \text{if } z = \infty \end{cases}$$

For each given z, such a basis is **easy to compute**.

Exponential parts of local solutions

A general local solution:

$$y(\tilde{x}) = \tilde{x}^{\alpha} \exp\left(u(\tilde{x}^{-1/s})\right) \sum_{k=0}^{m} \log(\tilde{x})^k b_k(\tilde{x}^{1/s})$$

- $\alpha \in C$
- u a polynomial, $u(0) = 0$
- $s \in \mathbb{N}$
- b_k a power series

The (local) **exponential part** of y is the factor $\tilde{x}^{\alpha} \exp(u(\tilde{x}^{-1/s}))$, modulo integer shifts of α.

Combining exponential parts

At each singular point $z_1, \ldots, z_n \in C \cup \{\infty\}$, the exponential part of a hyperexponential solution h must match **exactly one** of the exponential parts among the local solutions.

Singularity z_i	Exponential part 1	Exponential part 2	...
$z_1 = 0$	$\sqrt{x}\exp\left(\frac{1}{x}\right)$	$\sqrt{x}$	
$z_2 = 1$	$\exp\left(\frac{1}{1-x}\right)$	1	
$z_3 = \infty$	$\sqrt{x}$	$\exp(x)$	

Example: $h = (1-x)\sqrt{x}\,\exp\left(\frac{1}{x}\right)$ is described by the tuple $(1, 2, 1)$

The combination problem

At each singular point z_i, denote the local exponential parts by $E_{i,1}, \ldots, E_{i,\ell_i}$, $\ell_i \leq r$.

Combination problem

Find all tuples $\boldsymbol{j} = (j_1, \ldots, j_n)$ such that $E_{1,j_1}, \ldots, E_{n,j_n}$ are the exponential parts of a hyperexponential solution.

There can be at most r such tuples.

Brute force: r^n possibilities (exponential time).

Our algorithm

We use **analytic continuation** to reduce the combination problem to a **linear algebra problem**.

We have to test **only a polynomial number** of combinations.

A different approach to eliminating combinations (using modular techniques) is given by Cluzeau and van Hoeij (2004). They do not prove that their method leaves a polynomial number of combinations.

Reducing the problem to linear algebra

V_i: vector space of all **local** solutions at z_i
$V_{i,j} \subseteq V_i$: vector space of local solutions at z_i with exponential part $E_{i,j}$

W: vector space of all **global** solutions
$W_{i,j} \subseteq W$: vector space of global solutions corresponding to $V_{i,j}$

Provided that we can map local solutions to global solutions, finding a combination amounts to **finding a vector space intersection**:

$$W_{1,j_1} \cap W_{2,j_2} \cap \cdots \cap W_{n,j_n} \neq \{0\}$$

Abstract linear algebra problem

Let W be a vector space of dimension r. We are given n decompositions of W as a direct sum of $\ell \leq r$ subspaces:

$$\begin{array}{ccccccccc} W_{1,1} & \oplus & W_{1,2} & \oplus & \dots & \oplus & W_{1,\ell} & = & W \\ W_{2,1} & \oplus & W_{2,2} & \oplus & \dots & \oplus & W_{2,\ell} & = & W \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ W_{n,1} & \oplus & W_{n,2} & \oplus & \dots & \oplus & W_{n,\ell} & = & W \end{array}$$

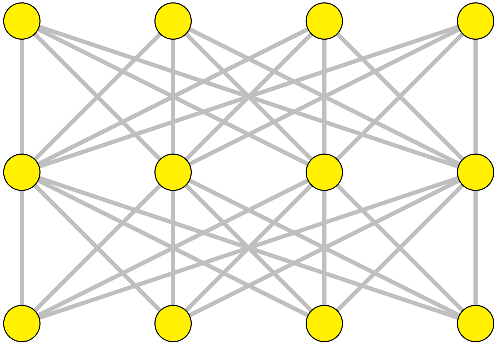
We want to find all tuples $\boldsymbol{j} = (j_1, \dots, j_n) \in \{1, \dots, \ell\}^n$ such that $W_{\boldsymbol{j}} := W_{1,j_1} \cap W_{2,j_2} \cap \cdots \cap W_{n,j_n} \neq \{0\}$.

Lemma

There are at most $\dim W = r$ *such tuples.*

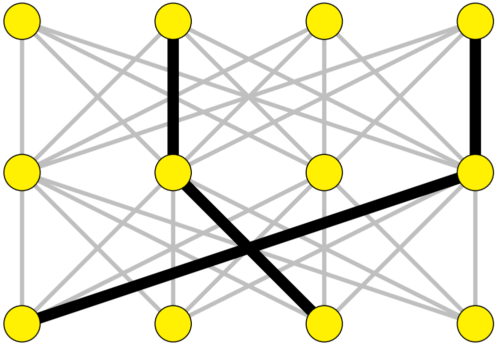
Finding all combinations

Exponential-time algorithm:



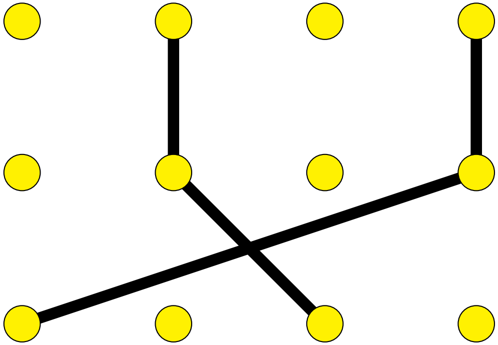
Finding all combinations

Exponential-time algorithm:



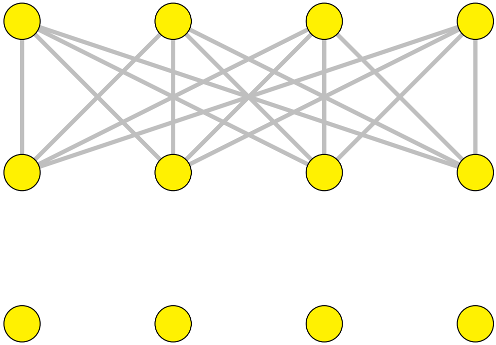
Finding all combinations

Exponential-time algorithm:



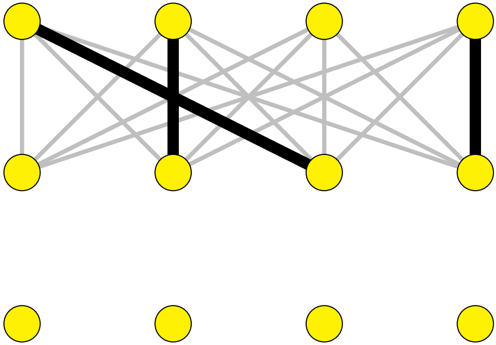
Finding all combinations

Polynomial-time algorithm:



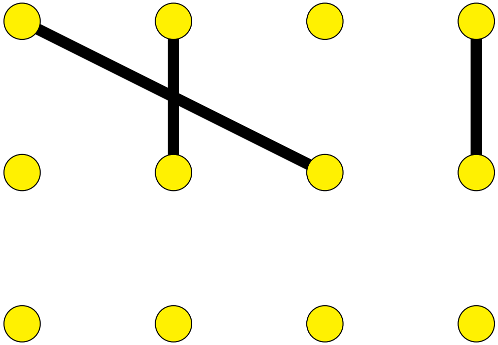
Finding all combinations

Polynomial-time algorithm:



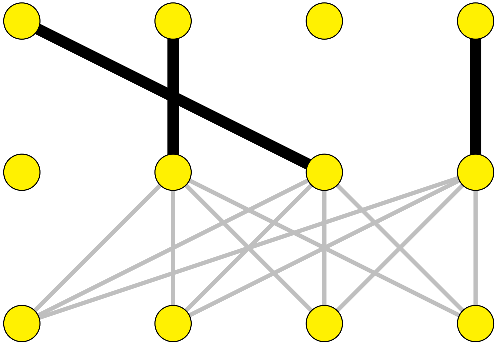
Finding all combinations

Polynomial-time algorithm:



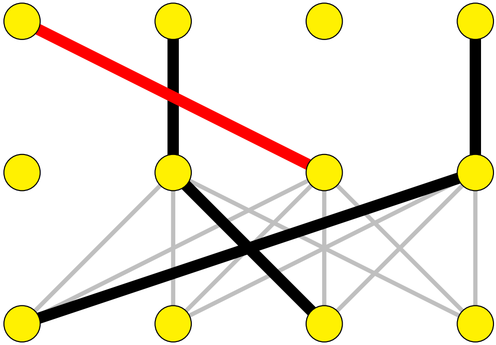
Finding all combinations

Polynomial-time algorithm:



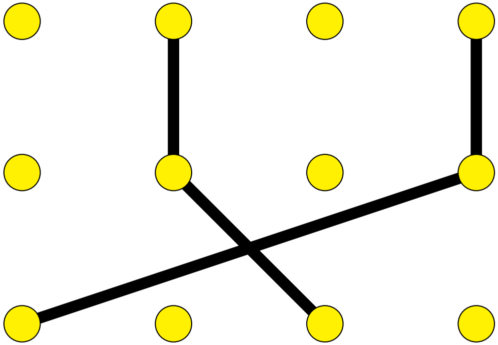
Finding all combinations

Polynomial-time algorithm:



Finding all combinations

Polynomial-time algorithm:



Complexity of the combination algorithm

Proposition

The combination algorithm performs no more than $O(nr^4)$ field operations. In particular, the number of operations is polynomial in both n and r.

It is easy to prove the weaker bound $O(nr^5)$: by the previously stated lemma, we perform Gaussian elimination at most $O(nr^2)$ times.

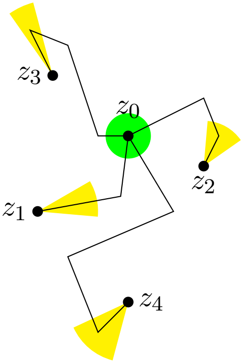
Mapping local solutions to global solutions

As the global solution space W, we take the solutions of $Py = 0$ in a neighborhood of some ordinary (nonsingular) point z_0.

A vector in W can be represented by the numerical values

$$\begin{bmatrix} y(z_0) \\ y'(z_0) \\ \vdots \\ y^{(r-1)}(z_0) \end{bmatrix}$$

Analytic continuation



We use **analytic continuation** to extend local solutions from each singular point $z_i, 1 \le i \le n$, to the common evaluation point z_0.

Technical considerations

Analytic continuation provides an appropriate linear map $V_i \to W$.

There are several technical points:

- Enough information is preserved
- We use resummation theory to interpret divergent power series as analytic functions
- We make some arbitrary but fixed choice of branch cuts

Numerical analytic continuation

We represent vectors in W by **numerical approximations**.

$\hat{y} \in V_{i,j}$: local solution at z_i

y: analytic continuation of $\hat{y}$

$Y_\varepsilon(z_0) \in \mathbb{Q}[i]^r$: numerical approximation of $Y = (y, y', \ldots, y^{(r-1)})$ at z_0

Using the algorithm of van der Hoeven for numerical evaluation of D-finite functions, for every $\hat{y}$ and every $\varepsilon > 0$, we can compute a $Y_\varepsilon(z_0)$ with

$$\|Y(z_0) - Y_\varepsilon(z_0)\| < \varepsilon.$$

Numerical linear algebra

- R: true rank of matrix
- $S \leq R$: computed rank using interval Gaussian elimination
- At sufficiently high precision, $S = R$.

Possible outcomes of numerical intersection test:

- If S is maximal, then there is certainly no intersection.
- If S is not maximal, then either:
  - There is an intersection, or
  - the precision is insufficient.

  We cannot tell, so we include the combination to be safe.

We **never incorrectly discard a combination**. At worst, too many combinations can get through. Possible strategy: restart with increased precision if there are more than $2r$ combinations.

Efficiency

We do not know the necessary precision in advance. In particular, we cannot guarantee that the complexity really is polynomial.

Heuristically, we expect a low numerical precision to be sufficient.

Numerical approximations are only used to determine which exponential parts to combine, **not** to compute output coefficients (which may be huge).

Future work

- Implement the algorithm and evaluate its efficiency in practice.
- Combine with other strategies (e.g. Cluzeau and van Hoeij) for eliminating combinations.
- Investigate using Miller's algorithm to discard divergent local solutions that cannot correspond to hyperexponential solutions.