Progress in arbitrary-precision ball arithmetic: numerical integration and faster arithmetic

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Introduction

Arb (`http://arblib.org`) – open source C library for arbitrary-precision ball arithmetic

Real numbers:

```
[3.14159265358979323846264643 +/- 4.03e-25]
```

Complex numbers:

```
[-0.200293 +/- 8.48e-7] + [0.979736 +/- 3.44e-7]*I
```

Features: polynomials, matrices, special functions, . . .

High-level interfaces: SageMath, Nemo.jl, Python-FLINT, . . .

Rigorous numerical integration in Arb

Example: $\int_0^1 \cos(x)\sin(x)dx$

```
>>> from flint import *
>>> ctx.prec = 100
>>> acb.integral(lambda x, _: x.cos()*x.sin(), 0, 1)
[0.35403670913678559674939205737 +/- 8.68e-30]
```

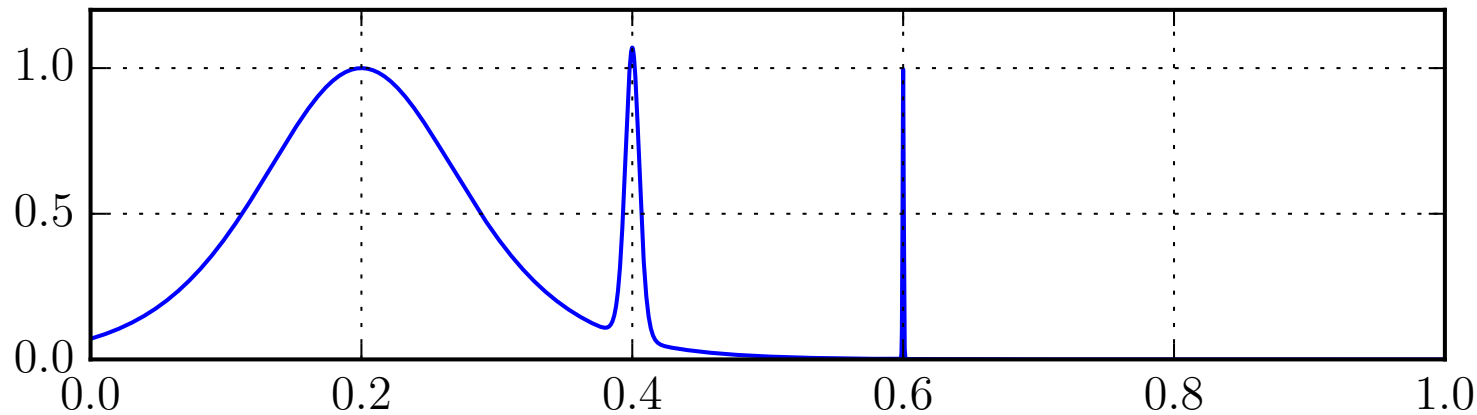
F. Johansson. *Numerical integration in arbitrary-precision ball arithmetic.* ICMS 2018. `https://arxiv.org/abs/1802.07942`

Documentation: `http://arblib.org/acb_calc.html`

Demo: `examples/integrals.c`

A nice and smooth function (Cranley and Patterson, 1971)

$$\int_0^1 \operatorname{sech}^2(10(x-0.2)) + \operatorname{sech}^4(100(x-0.4)) + \operatorname{sech}^6(1000(x-0.6))\ dx$$



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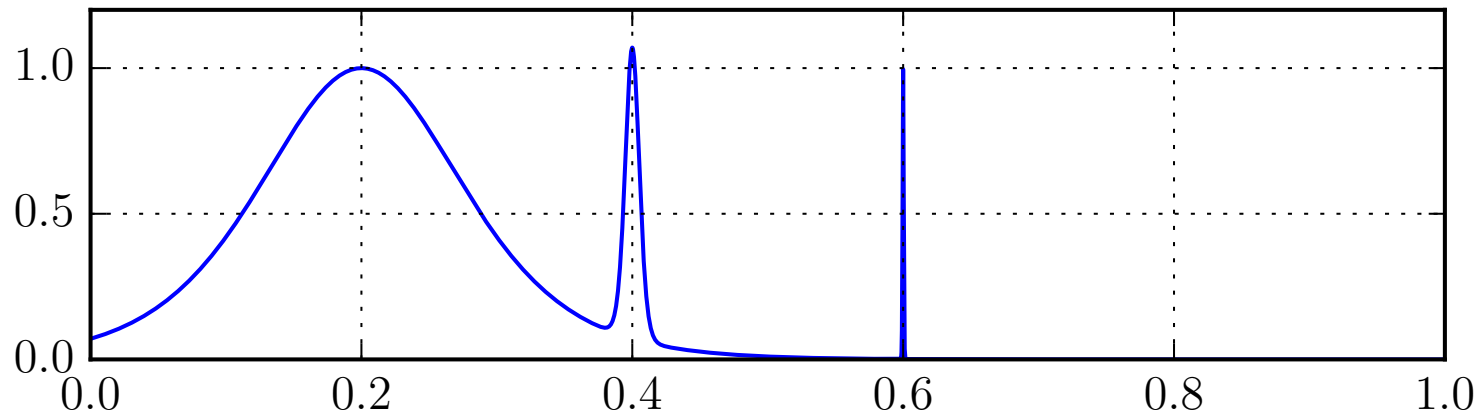

Mathematica `NIntegrate`: 0.209736

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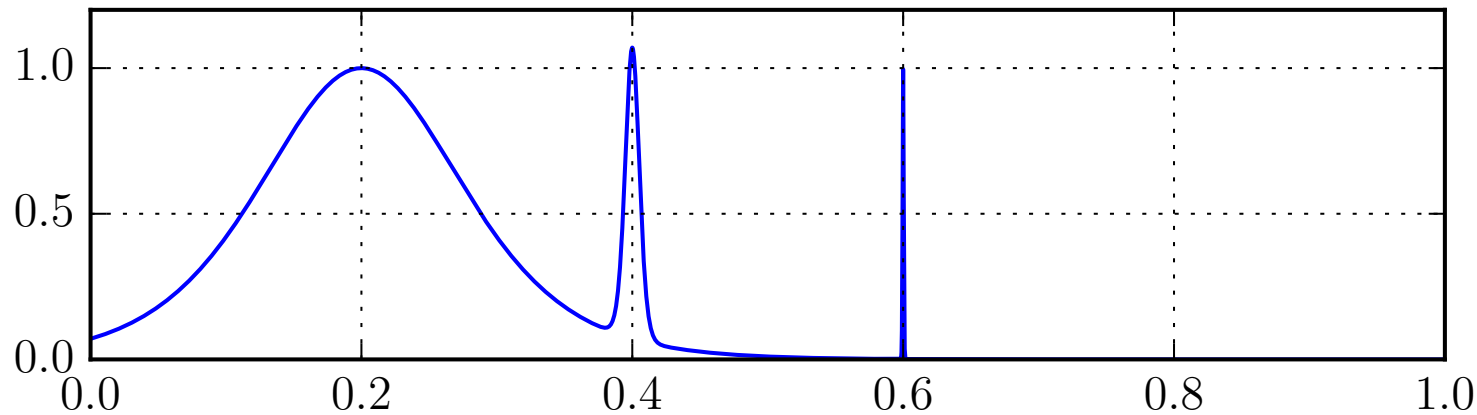

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Octave `quad`: 0.209736, error estimate 10^{-9}

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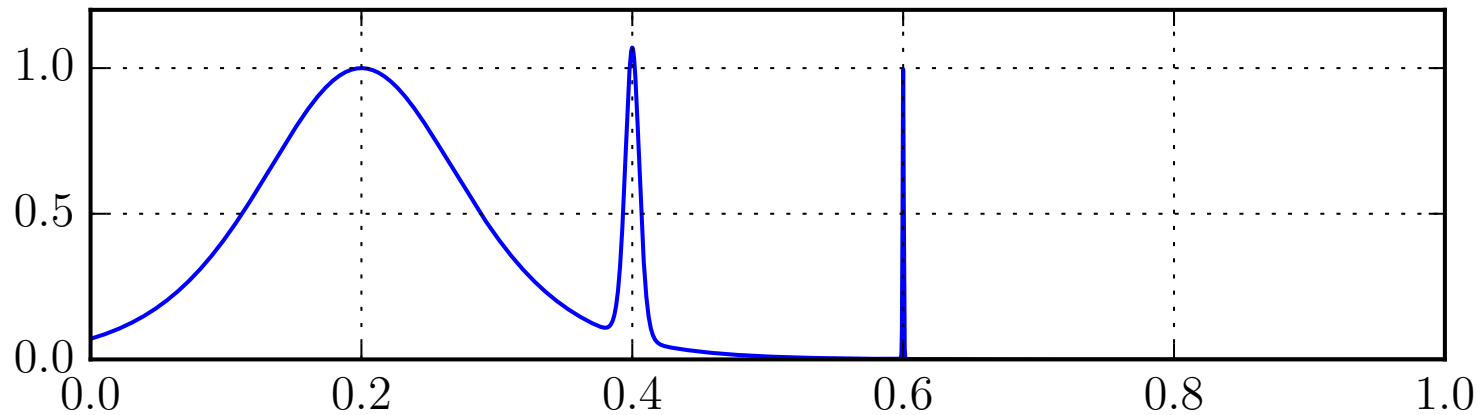

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Sage `numerical_integral`:	0.209736, error estimate 10^{-14}

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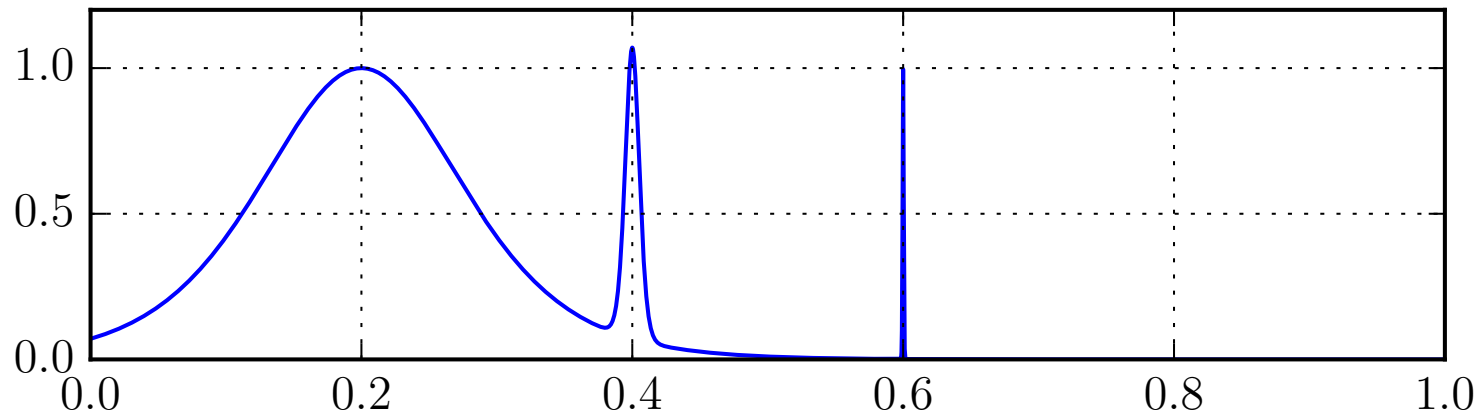

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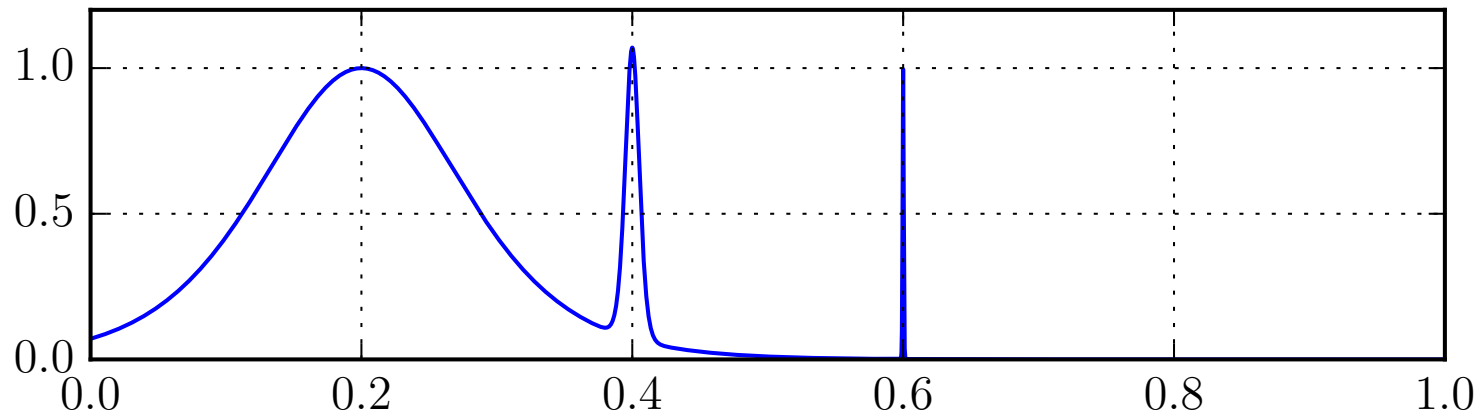

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mpmath `quad`:	0.209819

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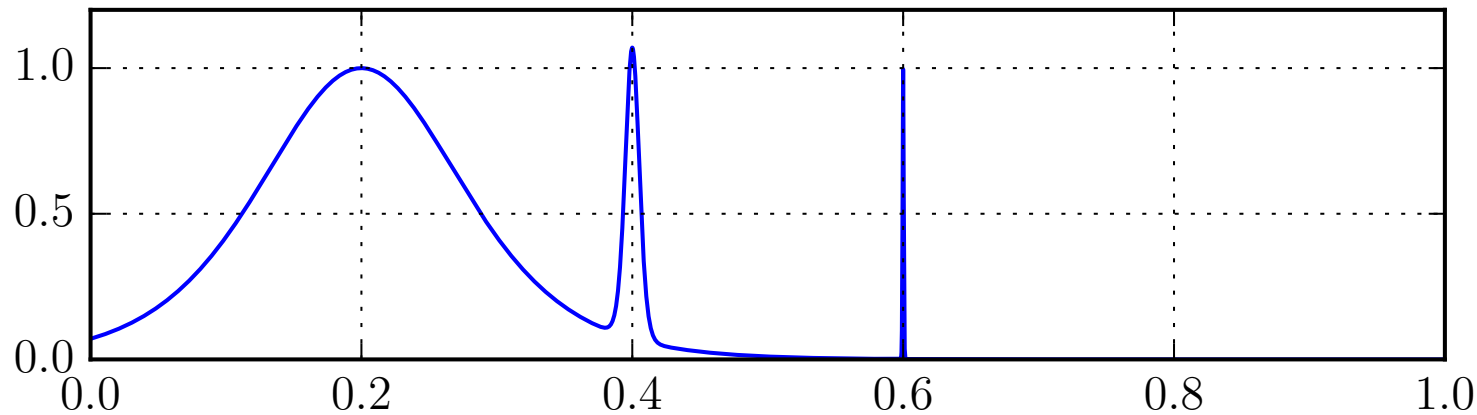

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Pari/GP `intnum`:	0.211316

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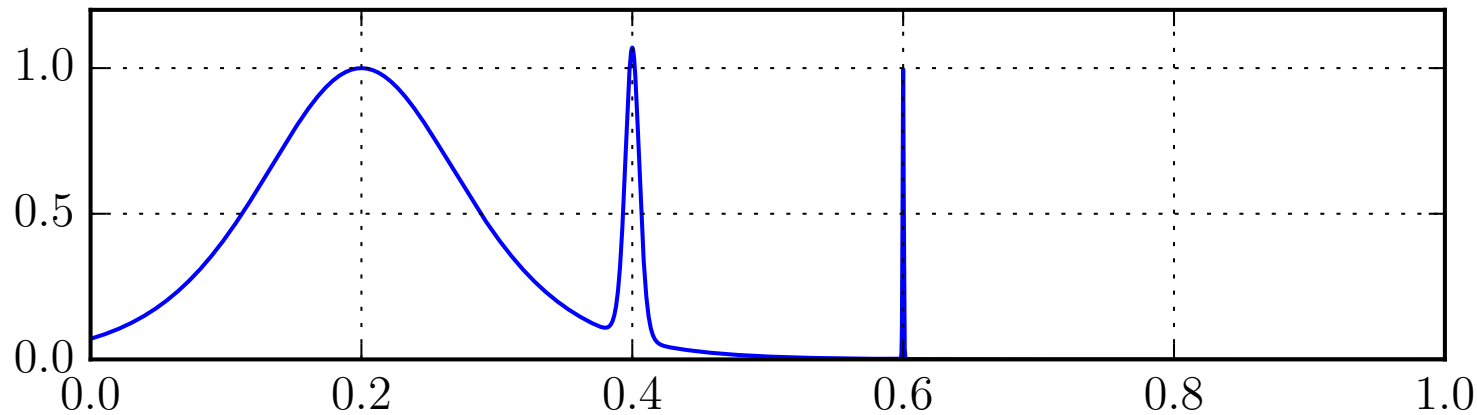

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Pari/GP `intnum`:	0.211316
Actual value:	0.210803

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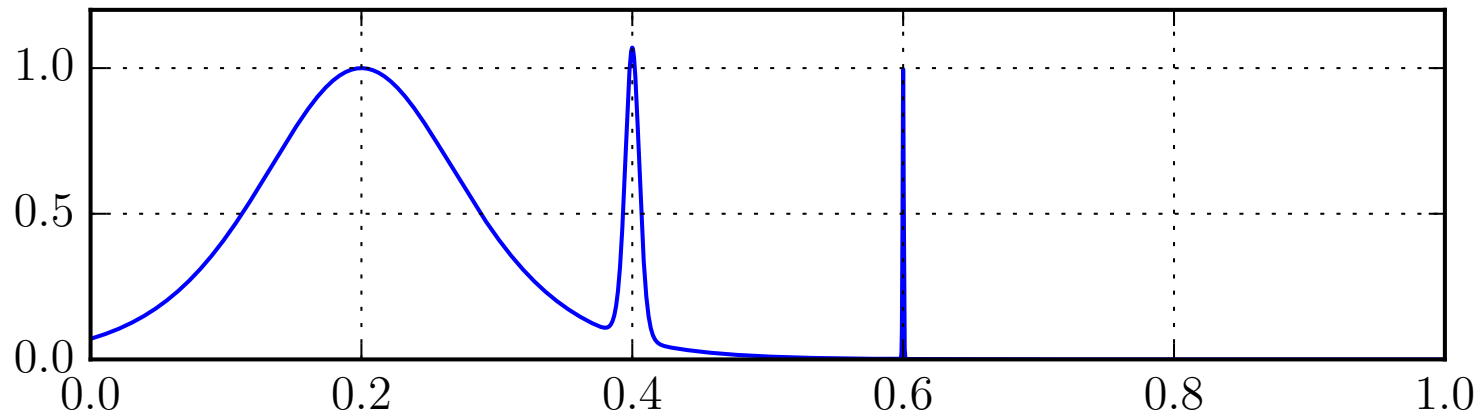


Arb, 64-bit precision:

```
[0.21080273550054928 +/- 4.55e-18]       # time 0.003 s
```

333-bit precision:

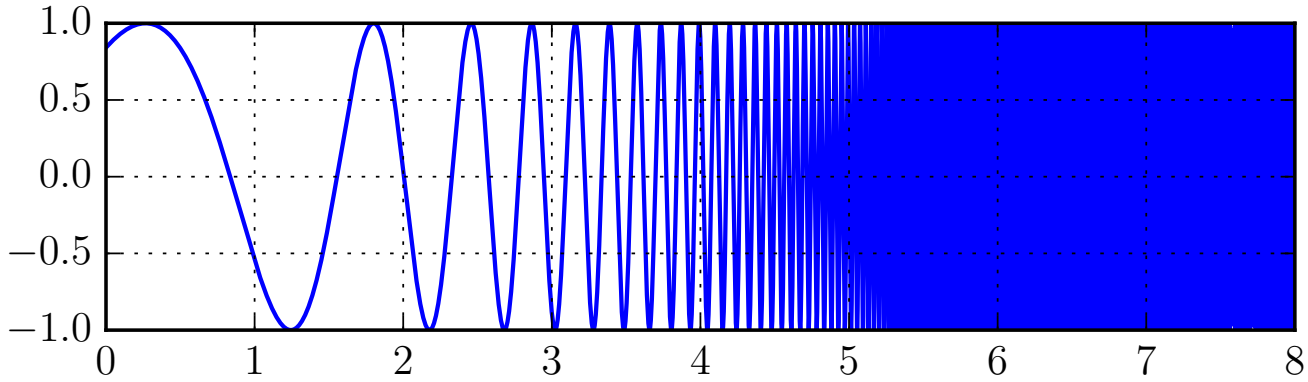
```
[0.21080273550054927737 56... +/- 3.67e-99]     # 0.02 s
```

3333-bit precision:

```
[0.21080273550054927737 56... +/- 1.39e-1001]  # 5.3 s
```

Another example: violent oscillation

$$\int_0^8 \sin(x+e^x)\,dx$$



S. Rump (2010) noticed that MATLAB's `quad` returned the incorrect `0.2511` after 1 second of computation.

Rump's INTLAB gives `[0.34740016, 0.34740018]` in about 1 s

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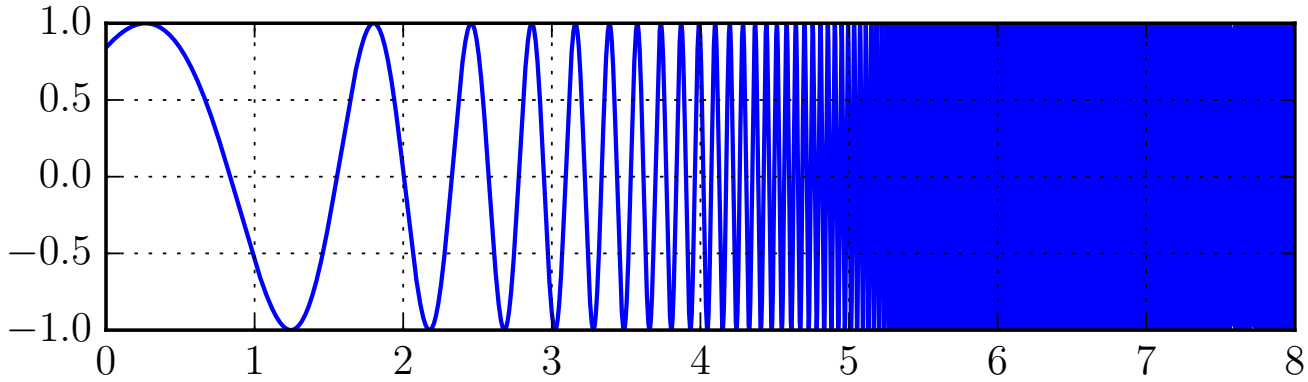

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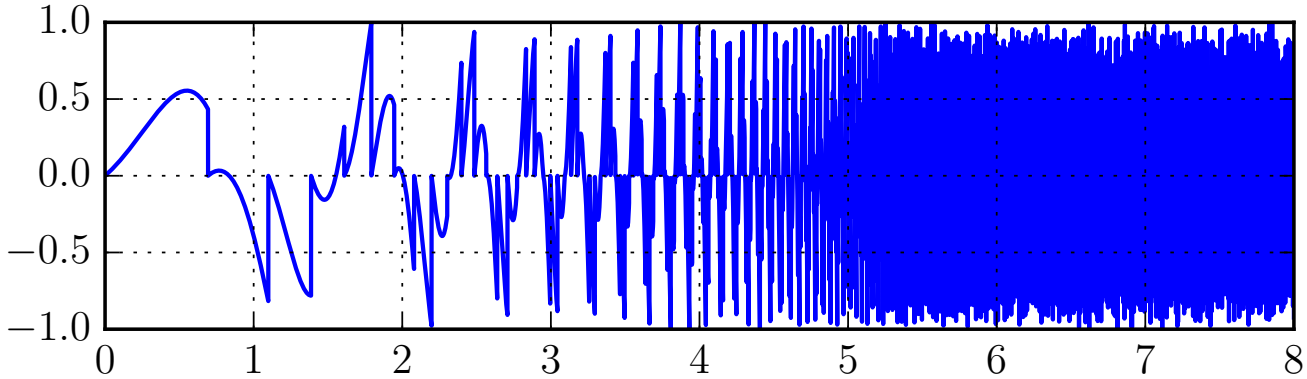
Arb at 64, 333, and 3333 bits:

```
[0.34740017265725 +/- 3.34e-15]    # 0.004 s
[0.34740017265... +/- 5.31e-96]    # 0.01 s
[0.34740017265... +/- 2.41e-999]   # 1 s
```

Yet another example: a monster

$$\int_0^8 (e^x - \lfloor e^x \rfloor) \sin(x+e^x)\,dx$$ – now with 2979 discontinuities!



64-bit precision:

```
[+/- 5.45e+3]                                  # time 0.14 s
```

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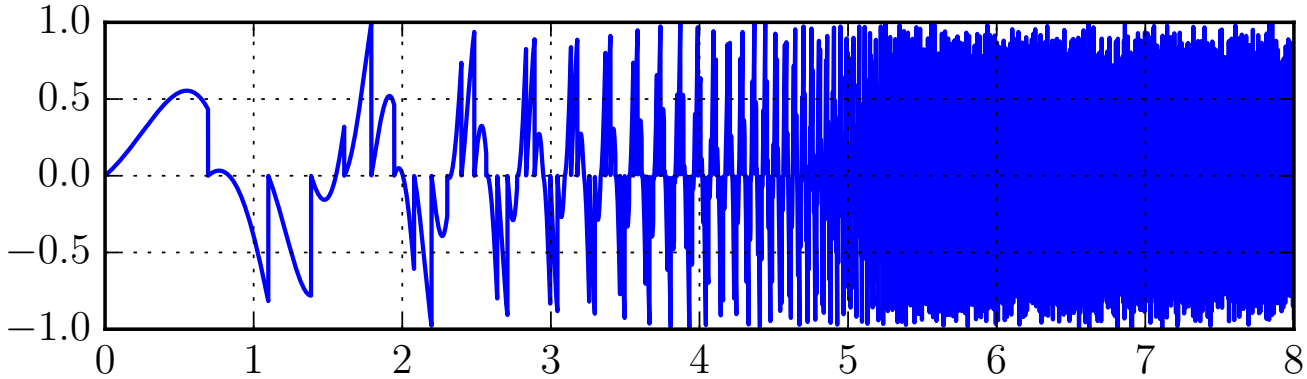


64-bit precision:

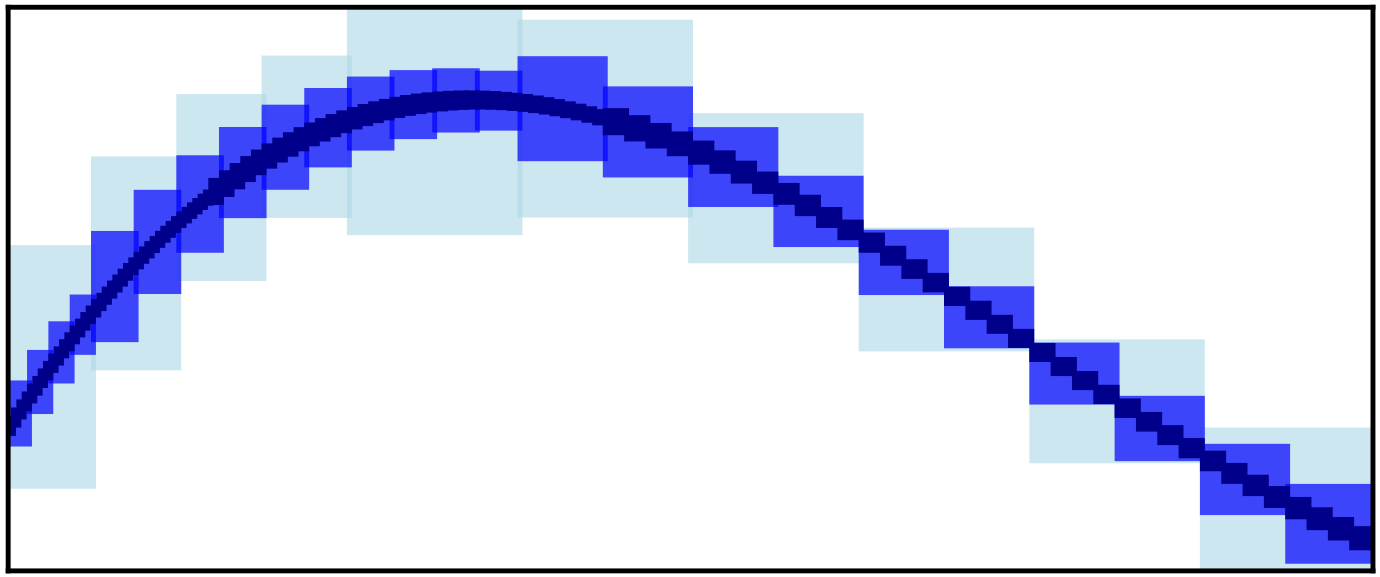
```
[+/- 5.45e+3]                                  # time 0.14 s
[0.0986517044784 +/- 4.46e-14]                 # time 5 s
```

333-bit precision:

```
[0.098651704478365206119658249764859856504169620792384491451091906830826680482290609839624064582
4 +/- 6.28e-95]  # 268 s
```

Brute force interval integration

$$\int_a^b f(x)dx \in (b-a)f([a,b]) \quad + \quad \text{adaptive subdivision of } [a,b]$$



This is simple and general, but we need $2^{O(p)}$ evaluations to achieve p-bit accuracy!

Efficient integration of analytic functions

We can achieve p-bit accuracy with $n = O(p)$ work.

Approximation:

$O(x^n)$ Taylor series

$$\int \sum_{k=0}^{n-1} a_k x^k = \sum_{k=0}^{n-1} a_k \frac{x^{k+1}}{k+1}$$

n-point quadrature

$$\int f(x)dx \approx \sum_{k=1}^{n} w_k f(x_k)$$

Error bounds:

Using derivatives $f^{(n)}$ on $[a, b]$

Using $|f|$ on a complex domain around $[a, b]$

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Fast generation of Gauss-Legendre quadrature nodes (x_k, w_k):

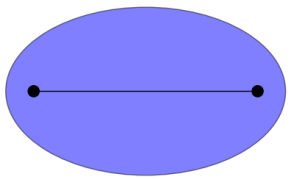
F.J. and M. Mezzarobba, SIAM J. Sci. Comp. 2018, `arxiv.org/abs/1802.03948`

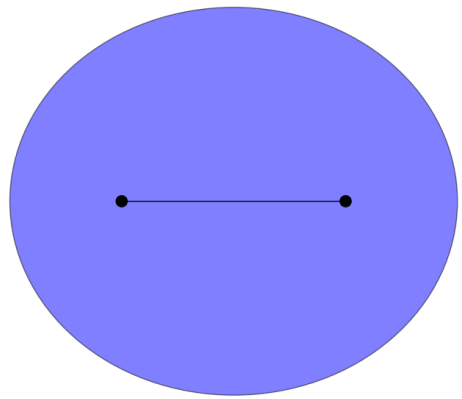
1-3 s (instead of 1 min) precomputation for 1000 digits

Error bounds for Gauss-Legendre quadrature

If f is analytic with $|f(z)| \leq M$ on an ellipse E with foci $-1, 1$ and semi-axes X, Y with $\rho = X + Y > 1$, then

$$\left| \int_{-1}^{1} f(x)\,dx - \sum_{k=1}^{n} w_k f(x_k) \right| \leq \frac{M}{\rho^{2n}} \cdot C_\rho$$





$X = 1.25, Y = 0.75, \rho = 2.00$ $\qquad$ $X = 2.00, Y = 1.73, \rho = 3.73$

Fast convergence when no singularities are close to $[a, b]$, but should be combined with subdivision otherwise!

Adaptive integration algorithm

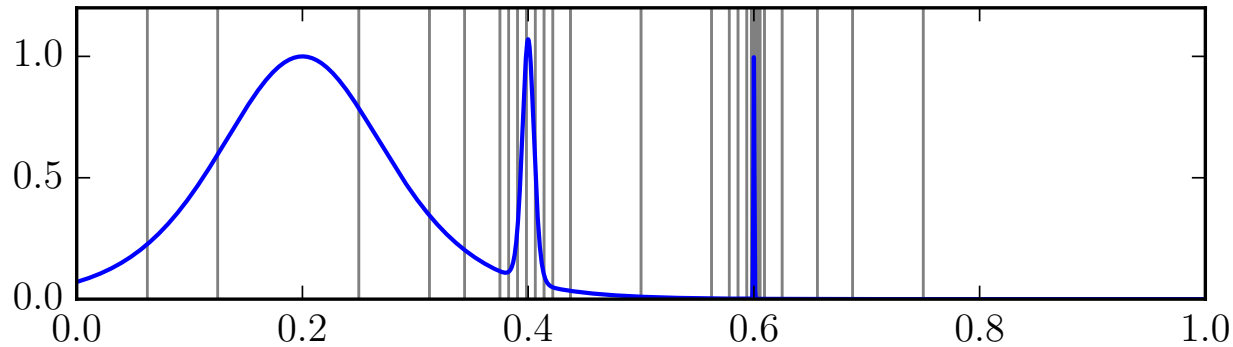
1. Compute $(b-a)f([a,b])$. If the error is $\leq \varepsilon$, done!
2. On an ellipse E around $[a,b]$, bound $|f|$ and check that f is analytic. If the error of Gauss-Legendre quadrature is $\leq \varepsilon$, compute it – done!
3. Split at $m = (a+b)/2$ and integrate on $[a,m]$, $[m,b]$ recursively.

Knut Petras (2002) pointed out that this guarantees rapid convergence for a large class of piecewise analytic functions.

Adaptive subdivision

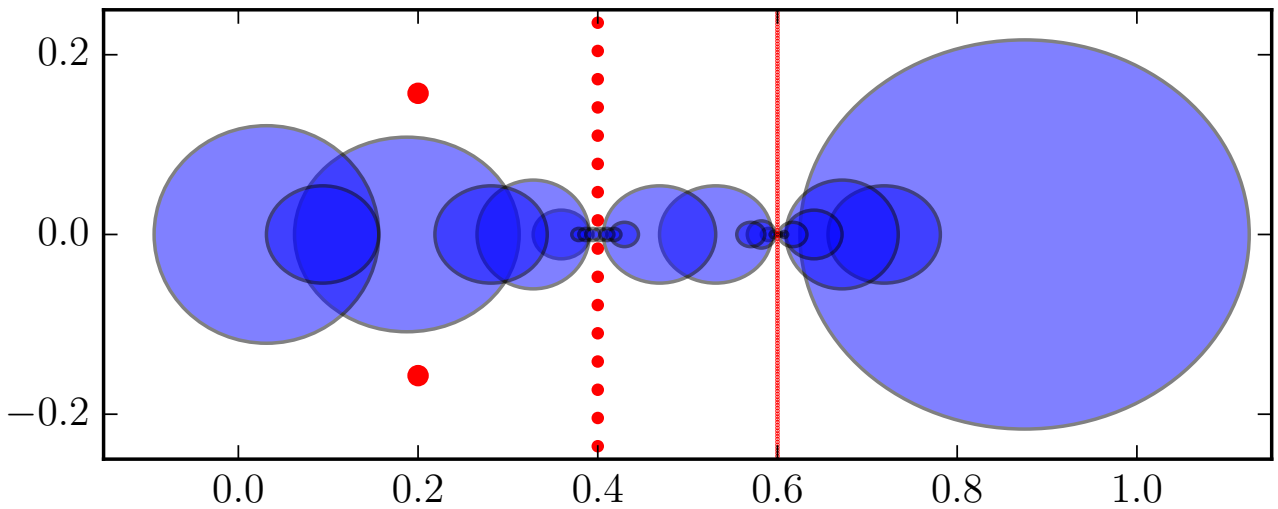
$$\int_0^1 \operatorname{sech}^2(10(x-0.2)) + \operatorname{sech}^4(100(x-0.4)) + \operatorname{sech}^6(1000(x-0.6))\ dx$$

Arb chooses
31 subintervals,
narrowest is 2^{-11}

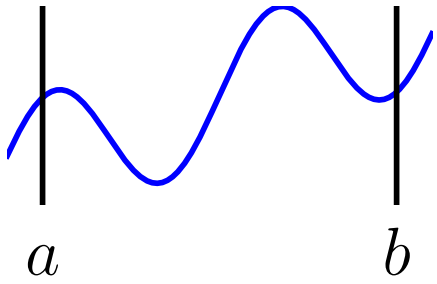


Complex ellipses
used for bounds

Red dots = poles

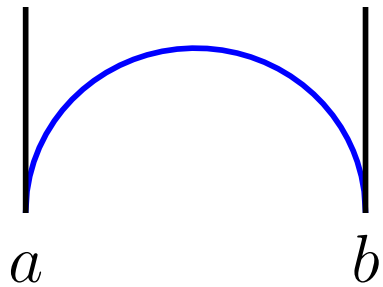


Typical proper integrals



Analytic around $[a, b]$

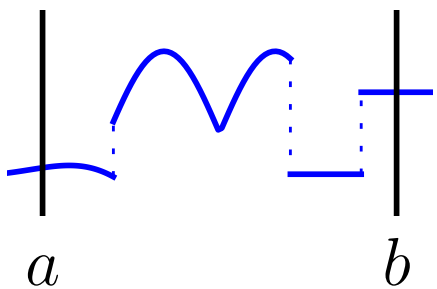
Complexity: $O(p)$



Bounded algebraic-type singularities

Example: $\sqrt{1-x^2}$

Complexity: $O(p^2)$



Piecewise analytic functions*

Examples: $\lfloor x \rfloor, \operatorname{sgn}(x), |x|, \max(f(x), g(x))$

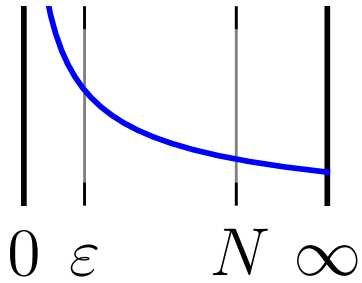
Complexity: $O(p^2)$

* Trick: extend piecewise real functions to the complex plane.

Discontinuities $\rightarrow$ branch cuts.

Typical improper integrals

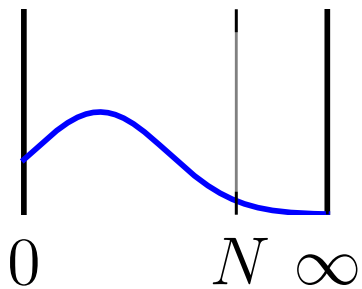
Manual truncation required, e.g. $\int_0^\infty f(x)dx \approx \int_\varepsilon^N f(x)dx$ if $|a|$, $|b|$ or $|f| \to \infty$



Algebraic blow-up or decay

Examples: $\int_0^1 \frac{dx}{\sqrt{x}}$, $\int_0^1 \log(x)dx$, $\int_0^\infty \frac{dx}{1+x^2}$

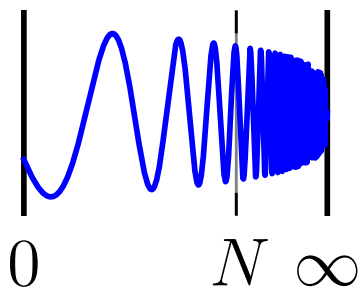
Complexity: $O(p^2)$



Exponential decay

Example: $e^{-x}\sin(x)$

Complexity: $O(p \log p)$



Essential singularity with slow decay

Example: $\int_1^\infty \frac{\sin(x)}{x}dx$

Complexity: $2^{O(p)}$

Timings: f analytic around $[a, b]$

p	Pari/GP	mpmath	Arb	Sub	Eval	Pari/GP	mpmath	Arb	Sub	Eval
	$I_0 = \int_0^1 1/(1+x^2)dx$					$I_1 = \int_0^1 \sum_{k=1}^{3} \operatorname{sech}^{2k}(10^k(x-0.2k))\,dx$				
64	0.00039	0.0011	0.000036	2	52	0.54	5.0	0.0031	31	768
333	0.0043	0.0058	0.00017	2	188	12	38	0.023	31	3086
3333	1.0	0.13	0.012	2	2056	3385	-	5.3	31	30092
	$I_2 = \int_0^\pi x\sin(x)/(1+\cos^2(x))dx$					$I_3 = \int_0^{1000} W_0(x)dx$				
64	0.00077	0.0046	0.00022	6	159	0.0037	0.032	0.00093	12	273
333	0.0088	0.037	0.0018	6	643	0.052	0.25	0.0095	12	1109
3333	2.2	4.4	0.43	6	6171	11	25	1.3	12	12043
	$I_4 = \int_0^{100} \sin(x)dx$					$I_5 = \int_0^8 \sin(x+e^x)\,dx$				
64	0.0012	0.0014	0.000070	1	72	0.063	0.25	0.0035	25	2239
333	0.015	0.018	0.00029	1	139	0.22	0.58	0.013	21	3940
3333	2.0	0.71	0.031	1	526	14	12	0.9	6	8341
	$I_6 = \int_{-1}^1 e^{-x}\operatorname{erf}\left(\sqrt{1250}\,x+\frac{3}{2}\right)dx$					$I_7 = \int_1^{1+1000i} \Gamma(x)dx$				
64	0.024	0.057	0.0054	6	438	0.054	0.093	0.0046	12	324
333	0.50	0.22	0.047	4	791	0.65	1.1	0.091	14	1456
3333	173	466	5.7	2	2923	561	847	48	14	16535

Timings: endpoint singularities and infinite intervals

p	Pari/GP	mpmath	Arb	Sub	Eval	Pari/GP	mpmath	Arb	Sub	Eval
	$E_0 = \int_0^1 \sqrt{1-x^2}\,dx$					$E_1 = \int_0^\infty 1/(1+x^2)\,dx$ *				
64	0.00041	0.00067	0.00057	44	674	0.00060	0.0012	0.0022	190	2887
333	0.0044	0.0060	0.015	223	12687	0.0068	0.011	0.048	997	51900
3333	0.94	0.18	6.6	2223	1.2 M	1.7	0.24	27	9997	4.7 M
	$E_2 = \int_0^1 \log(x)/(1+x)\,dx$ *					$E_3 = \int_0^\infty \operatorname{sech}(x)\,dx$ *				
64	0.00081	0.00094	0.0012	67	1026	0.0011	0.0043	0.00022	7	181
333	0.011	0.011	0.038	336	19254	0.013	0.098	0.0019	9	853
3333	1.7	1.08	106	3336	1.8 M	3.5	3.3	0.68	12	12046
	$E_4 = \int_0^\infty e^{-x^2+ix}\,dx$ *					$E_5 = \int_0^\infty e^{-x}\,\mathrm{Ai}(-x)\,dx$ *				
64	0.0014	0.016	0.00017	1	98	-	0.91	0.012	9	842
333	0.017	0.13	0.0016	2	397	-	26	0.94	124	24548
3333	4.7	7.1	0.47	4	3894	-	10167	502	1205	0.7 M

* For Arb, the path was truncated manually (with error $\leq 2^{-p}$)

Timings: mid-interval jumps/kinks

p	Arb	Sub	Eval	Arb	Sub	Eval
	$\int_0^1 \lvert x^4+10x^3+19x^2-6x-6\rvert\, e^x\, dx$			$\int_0^{100} \lceil x \rceil\, dx$		
64	0.0016	70	1093	0.014	5536	16606
333	0.049	339	18137	0.12	33512	100534
3333	101	3339	1624951	1.6	345512	1036534
	$\int_0^{10}(x-\lfloor x\rfloor-\frac{1}{2})\max(\sin(x),\cos(x))dx$			$\int_{-1-i}^{-1+i} \sqrt{x}\, dx$		
64	0.026	1257	16168	0.0021	132	1462
333	1.2	7076	394881	0.067	670	28304
3333	2588	71984	39128525	35	6670	2669940

High accuracy with mpmath or Pari/GP is not possible without manually splitting at all the singular points.

Defining functions

The user provides the integrand $f(z)$ as a black-box function that takes two parameters:

- A complex ball (rectangle) representing z
- A boolean flag *analytic*
  - *False* - the function returns an enclosure of $f(z)$.
    There are no assumptions about analyticity.
  - *True* - the function returns an enclosure of $f(z)$.
    It must return non-finite (NaN, $[\pm\infty]$) if the ball z contains any non-analytic point of f.

The user can always ignore the *analytic* flag when f is a composition of meromorphic functions.

Defining functions

The *analytic* flag **must** be handled

when the integrand has branch cuts.

$$\int_1^4 \sqrt{x}\, dx = \frac{14}{3}$$

```
>>> F1 = lambda x, _: x.sqrt()                    # WRONG!
>>> acb.integral(F1, 1, 4)
[4.66941489478101 +/- 7.48e-15]

>>> F2 = lambda x, a: x.sqrt(analytic=a)          # correct
>>> acb.integral(F2, 1, 4)
[4.666666666667 +/- 4.62e-14]
```

Versions of common functions ($\sqrt{x}$, $\log(x)$, x^y, $|x|$, $\lfloor x \rfloor$, $\max(x,y)$, ...) with builtin branch cut detection are provided in Arb.

Optional settings for the integration algorithm

The user specifies:

- Working precision p
- Absolute and relative tolerances ε_{abs} and ε_{rel}

Configurable work limits:

- Maximum quadrature degree (default: $O(p)$)
- Number of calls to the integrand (default: $O(p^2)$)
- Number of queued subintervals (default: $O(p)$)
- Use stack (default) or global priority queue for the list of subintervals generated by bisection

Applications

- Special functions:

$$\Gamma(s,z) = \int_z^\infty t^{s-1} e^{-t}\, dt$$

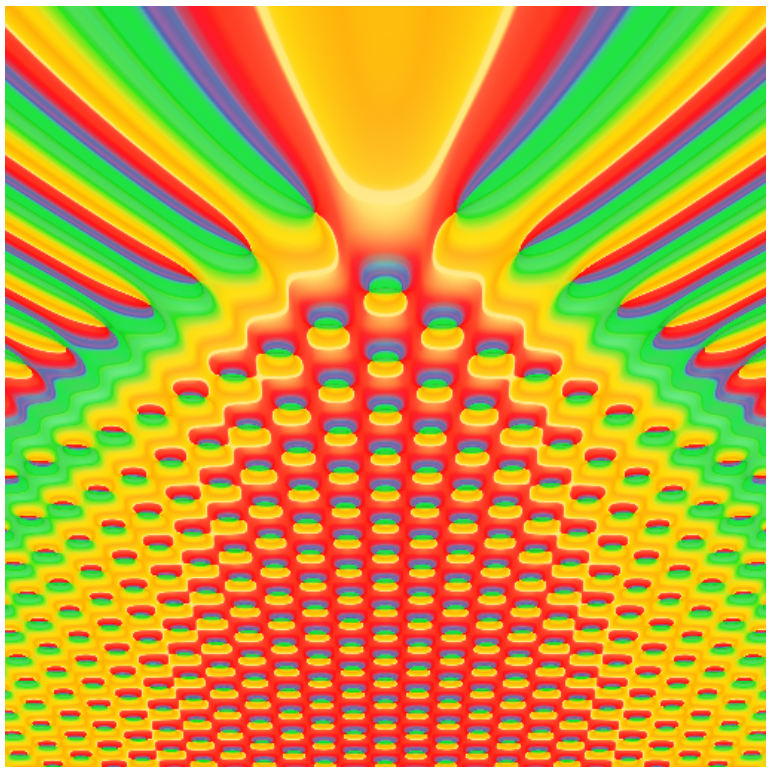
- (Inverse) Laplace/Fourier/Mellin transforms
- Taylor/Laurent/Fourier coefficients
- Counting zeros and poles:

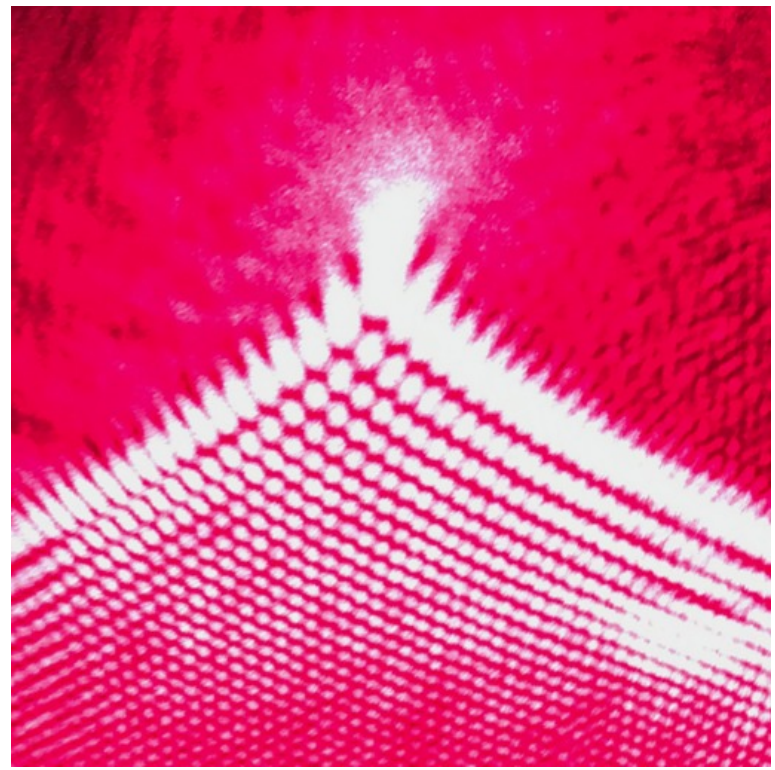
$$N - P = \frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)}\, dz$$

- Acceleration of series (Euler-Maclaurin summation. . .)

Example: diffraction catastrophe integrals

$$P(x,y) = \int_{-\infty}^{\infty} e^{i(t^4+yt^2+xt)}\,dt = 2\int_{0}^{\infty} e^{-t^4+at^2+b}\cosh(ct)\,dt$$





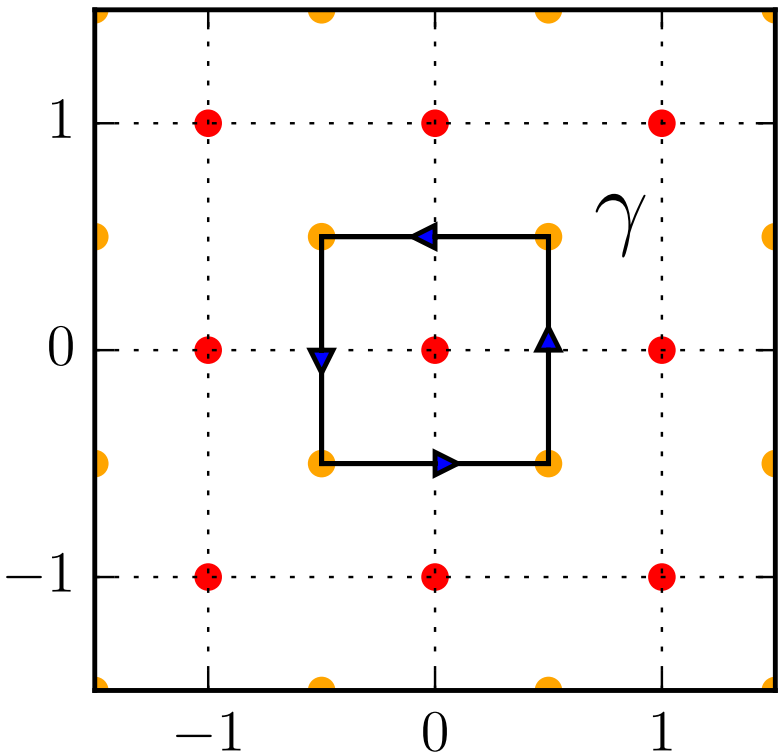
Left: 512×512 image rendered in 15 minutes with Arb ($|x| \leq 12.5$, $-20 \leq y \leq 5$). Using doubling precision (30, 60, ... bits). Near the bottom, $p = 120$ is required.

Right: photo of a cusp caustic produced by illuminating a flat surface with a laser beam through a droplet of water (image credit: Dan Piponi, CC-BY-SA)

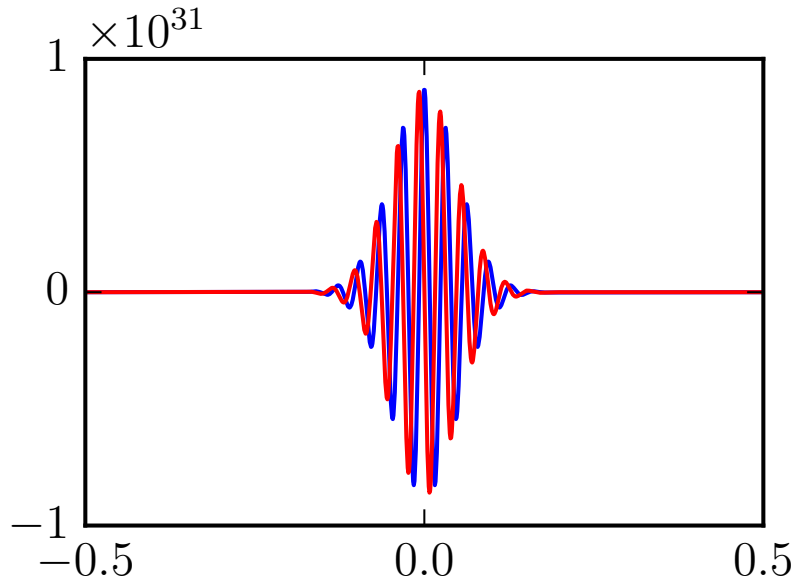
Example: Laurent series of elliptic functions

$$\wp(z;\tau) = \sum_{n=-2}^{\infty} a_n(\tau) z^n, \quad a_n = \frac{1}{2\pi i} \int_\gamma \frac{\wp(z)}{z^{n+1}} dz$$

$\wp(z)$ with $\tau = i$ has poles at $z = M + Ni \quad (M, N \in \mathbb{Z})$.



One segment ($n = 100$):



Example: Laurent series of elliptic functions

$$\wp(z;\tau) = \sum_{n=-2}^{\infty} a_n(\tau) z^n, \quad a_n = \frac{1}{2\pi i} \int_\gamma \frac{\wp(z)}{z^{n+1}} dz$$

$a_{-2}, \ldots, a_{100}$ with 333-bit precision (0.8 seconds for each a_n):

```
a[-2]  = [1.0000000000000000 ... 00000 +/- 3.57e-98] + [+/- 1.89e-98]*I
a[-1]  =                                [+/- 4.11e-98] + [+/- 2.57e-98]*I
a[0]   =                                [+/- 1.02e-97] + [+/- 5.39e-98]*I
a[1]   =                                [+/- 1.41e-97] + [+/- 1.35e-97]*I
a[2]   = [9.453636006461692  ... 52235 +/- 4.44e-97] + [+/- 2.48e-97]*I
a[3]   =                                [+/- 4.47e-97] + [+/- 4.60e-97]*I
...
a[94]  = [380.00000000000135 ... 63746 +/- 9.24e-70] + [+/- 8.27e-70]*I
a[95]  =                                [+/- 1.37e-69] + [+/- 1.37e-69]*I
a[96]  =                                [+/- 2.93e-69] + [+/- 2.91e-69]*I
a[97]  =                                [+/- 5.81e-69] + [+/- 5.82e-69]*I
a[98]  = [395.9999999999996482...46383 +/- 2.90e-68] + [+/- 1.17e-68]*I
a[99]  =                                [+/- 2.32e-68] + [+/- 2.32e-68]*I
a[100] =                                [+/- 4.95e-68] + [+/- 4.95e-68]*I
```

Example: algorithm for the Stieltjes constants

With I. Blagouchine (`arxiv.org/abs/1804.01679`; to appear in Math. Comp.)

$$\zeta(s,v) = \sum_{k=0}^{\infty} \frac{1}{(k+v)^s} = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n(v)(s-1)^n$$

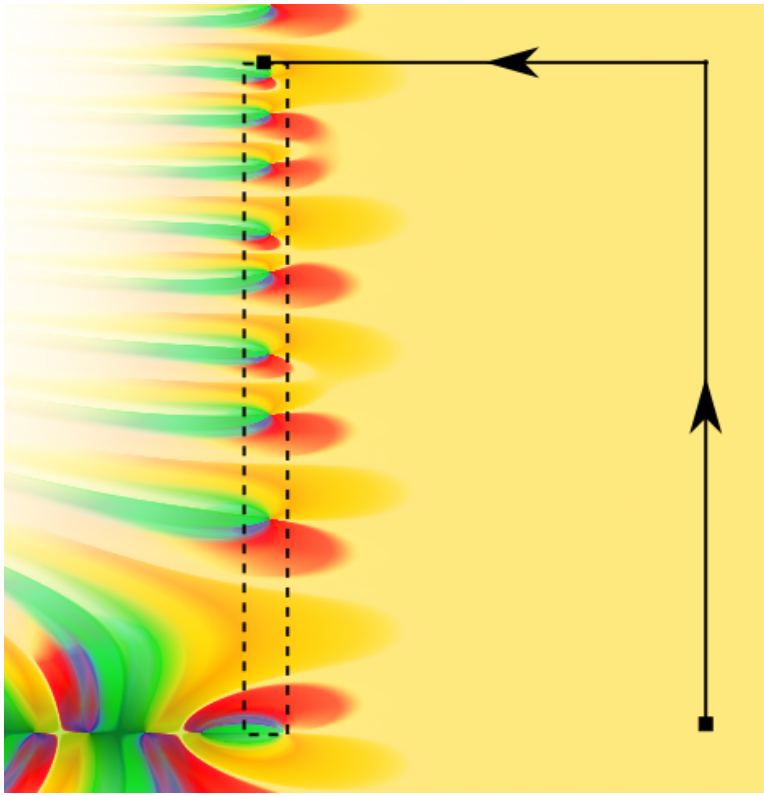
$$\gamma_n(v) = -\frac{\pi}{2(n+1)} \int_{-\infty}^{\infty} \frac{\left(\log\left(v - \frac{1}{2} + ix\right)\right)^{n+1}}{\cosh^2(\pi x)} dx$$

$\gamma_{10^{100}}(1) \in [3.18743141870239927999741646993 \pm 2.89 \cdot 10^{-30}] \cdot 10^{e}$

$e = 23463942922772540809493678383990911609034476898698373852057791115792156640521582344171254175433483694$

Some pen-and-paper analysis (steepest descent contour, tight enclosures near saddle point) needed for large n.

Example: zeros of the Riemann zeta function



Number of zeros of $\zeta(s)$ on $R = [0,1] + [0,T]i$:

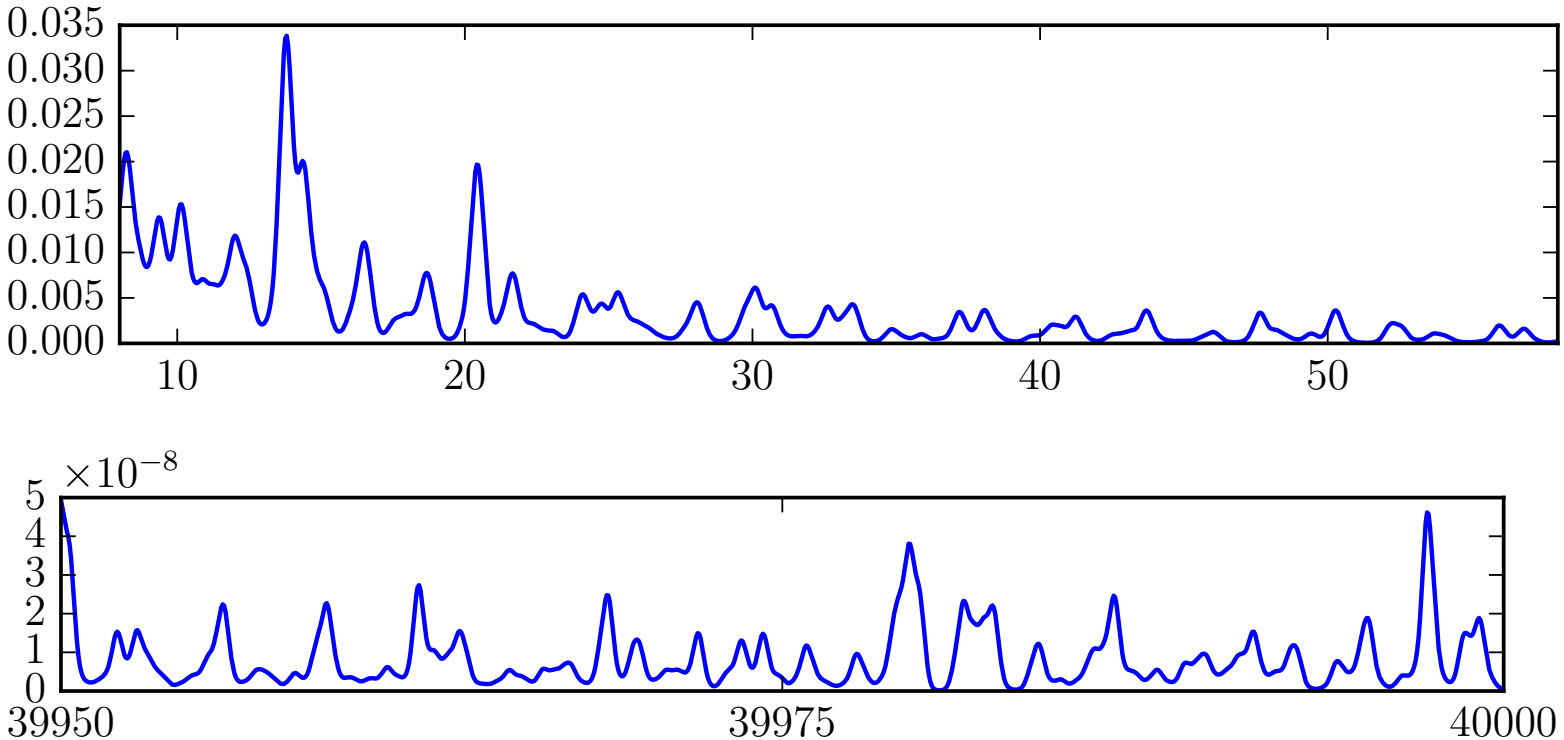
$$N(T) - 1 = \frac{1}{2\pi i}\int_\gamma \frac{\zeta'(s)}{\zeta(s)}ds = \frac{\theta(T)}{\pi} +$$

$$\frac{1}{\pi}\operatorname{Im}\left[\int_{1+\varepsilon}^{1+\varepsilon+Ti}\frac{\zeta'(s)}{\zeta(s)}ds + \int_{1+\varepsilon+Ti}^{\frac{1}{2}+Ti}\frac{\zeta'(s)}{\zeta(s)}ds\right]$$

T	p	Time (s)	Eval	Sub	$N(T)$
10^3	32	0.51	1219	109	`[649.00000 +/- 7.78e-6]`
10^6	32	16	5326	440	`[1747146.00 +/- 4.06e-3]`
10^9	48	1590	8070	677	`[2846548032.000 +/- 1.95e-4]`

Example: $|\zeta(s)|$-integrals (from Harald Helfgott)

$$\int_{-\frac{1}{4}+8i}^{-\frac{1}{4}+40000i} \left| \frac{F_{19}(s+\frac{1}{2})F_{19}(s+1)}{s^2} \right| |ds|, \quad F_N(s) = \zeta(s) \prod_{p \le N} (1-p^{-s})$$



We compute Taylor models $f(s) = g(s) + h(s)i + \varepsilon$ on subintervals $[a, a+0.5]$, and integrate $\sqrt{g^2(s)+h^2(s)}$. Total time: a few hours.

TODO for numerical integration

- Efficient and semi-automatic support for singularities, infinite intervals
  - User may specify scale, e.g. $|f(x)| \leq x^{\alpha} e^{\beta x^{\gamma}}$
  - Dedicated algorithms: Gauss-Jacobi, double exponential...
  - Algorithms for oscillatory integrals
- Symbolic interface
- Let user choose Taylor/Gauss-Legendre quadrature and bounds based on derivatives / complex magnitudes
- Better global adaptivity
- Many applications

Faster arbitrary-precision arithmetic

Teaser: time to multiply two 1000×1000 matrices

$p = 53$:

- OpenBLAS (1 thread): 0.066 s
- Julia BigFloat: 405 s (6 000 times slower)
- MPFR: 36 s (550 times slower)
- Arb: 3.6 s (55 times slower)

$p = 212$:

- QD: 111 s
- Julia BigFloat: 462 s
- MPFR: 110 s
- Arb: 8.2 s

This work

F. Johansson. *Faster arbitrary-precision dot product and matrix multiplication*. `https://arxiv.org/abs/1901.04289`

Separate cases:

- Small-N (dot products)
- Large-N (matrix multiplication)
- Approximate (no error bounds) and ball versions

Design constraints:

- True arbitrary precision
- Preserve entrywise information

$$\begin{pmatrix} [1.23 \cdot 10^{100} \pm 10^{80}] & -1.5 & 0 \\ 1 & [2.34 \pm 10^{-20}] & [3.45 \pm 10^{-50}] \\ 0 & 2 & [4.56 \cdot 10^{-100} \pm 10^{-130}] \end{pmatrix}$$

Dot product

$$\sum_{i=0}^{N-1} a_i b_i, \qquad a_i, b_i \in \mathbb{R} \text{ or } \mathbb{C}$$

In ball arithmetic: $\sum_{i=0}^{N-1} [m_i \pm r_i][m'_i \pm r'_i] = [m \pm r]$,

$$m = \sum_{i=0}^{N-1} m_i m'_i, \qquad r = \sum_{i=0}^{N-1} |m_i| r'_i + |m'_i| r_i + r_i r'_i$$

Kernel in basecase ($N \lesssim 10$ to 100) algorithms for:

- Matrix multiplication
- Triangular solving, recursive LU factorization
- Polynomial multiplication, division
- Power series arithmetic, transcendental functions

Dot product as an atomic operation

The old way:

```
arb_mul(s, a, b, prec);
for (i = 1; i < N; i++)
    arb_addmul(s, a + i, b + i, prec);
```

The new way:

```
arb_dot(s, NULL, 0, a, 1, b, 1, N, prec);
```

(More generally, computes $s = s_0 + (-1)^c \sum_{i=0}^{N-1} a_{i \cdot \text{astep}} b_{i \cdot \text{bstep}}$)

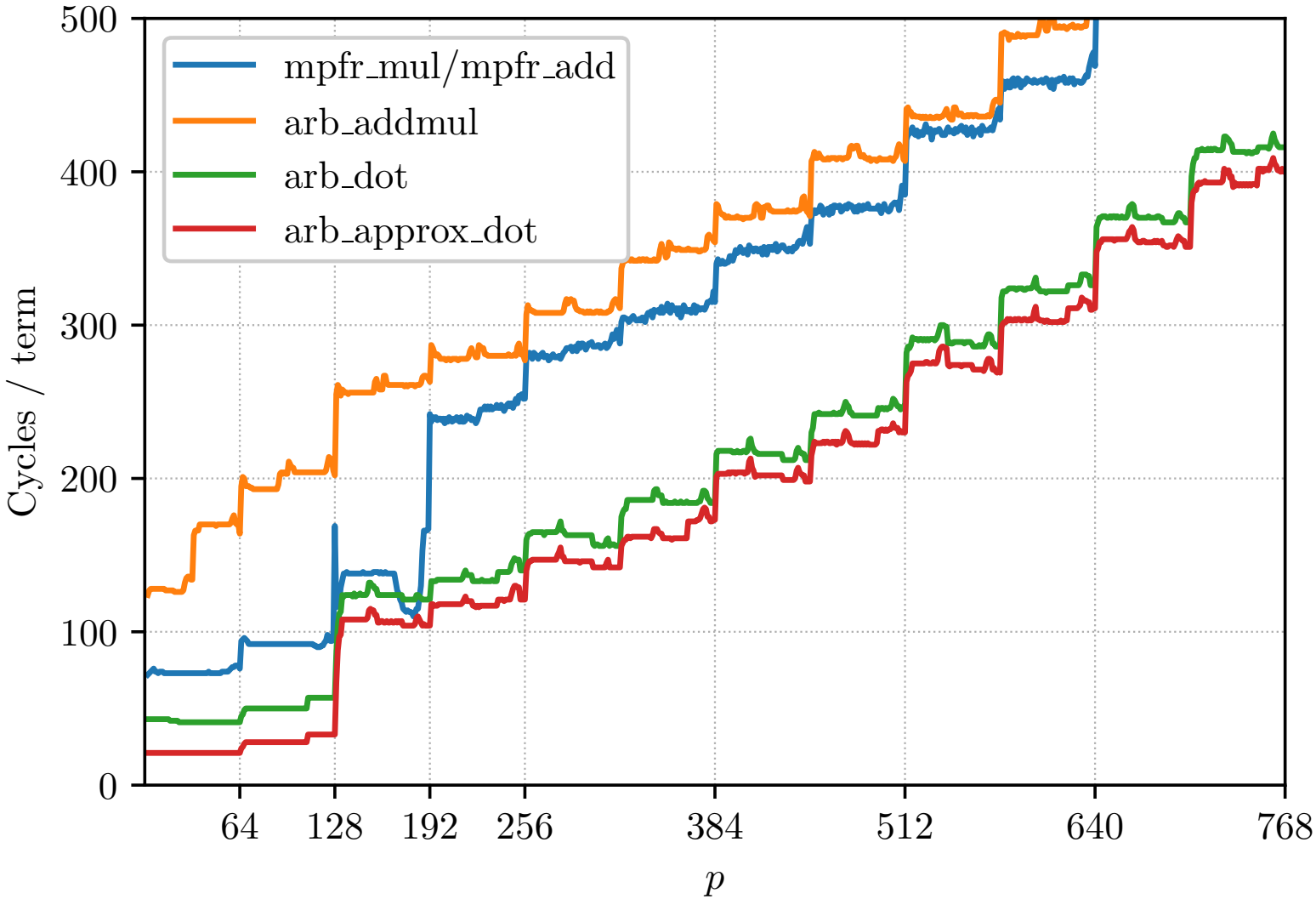
`arb_dot`, `acb_dot`, `arb_approx_dot`, `acb_approx_dot`

Dot product performance

Cycles/term to compute $\sum_i a_i b_i$ on an i5-4300U (Haswell)

	QD	MPFR (real)	Arb (real)		
p		mul/add	addmul	dot	approx
53		74	169	40	20
106	26	97	203	49	27
212	265	237	277	133	117
3392		4059	4875	3906	3880
13568		33529	39275	32476	32467
		MPC (complex)	Arb (complex)		
p		mul/add	addmul	dot	approx
53		570	772	166	84
106		885	911	208	112
212		1123	1346	555	478
3392		18527	17926	15691	15618
13568		129293	127672	125757	125634

Dot product performance



Outline of the algorithm

First pass: inspect all terms

- Detect any Inf/NaN or exponents requiring bignums (switch to fallback code)
- Count number of nonzero terms, bound exponents (separately for midpoint and radius dot products)

Second pass: compute the dot product!

- Use fixed-point accumulator (GMP `mpn` arithmetic), no intermediate normalization
- Single-word accumulators for radius dot product, arithmetic errors
- Single final rounding and conversion to a ball

Complex dot product $\simeq$ two length-$2N$ real dot products

Arbitrary-precision floating-point addition

Limbs are full 64-bit words

$a = 2^e \cdot$ |xxxxxxxx|xxxxxxxx|

$b = 2^f \cdot$ |xxxxxxxx|xxxxxxxx|xxxxxxxx|

Arbitrary-precision floating-point addition

Limbs are full 64-bit words

$a = 2^e \cdot$ |xxxxxxxx|xxxxxxxx|

$b = 2^f \cdot$ |xxxxxxxx|xxxxxxxx|xxxxxxxx|

The term with smaller exponent is right-shifted by $|f - e|$ bits, aligning the limb boundaries (several case distinctions)

```
                  |xxxxxxxx|xxxxxxxx|xxxxxxxx|
+                          |0000xxxx|xxxxxxxx|xxxx0000|
```

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Limbs are full 64-bit words

$a = 2^e \cdot$ |xxxxxxxx|xxxxxxxx|

$b = 2^f \cdot$ |xxxxxxxx|xxxxxxxx|xxxxxxxx|

The term with smaller exponent is right-shifted by $|f - e|$ bits, aligning the limb boundaries (several case distinctions)

```
                  |xxxxxxxx|xxxxxxxx|xxxxxxxx|
+                           |0000xxxx|xxxxxxxx|xxxx0000|

=  |0000000x|xxxxxxxx|xxxxxxxx|xxxxxxxx|xxxx0000|
```

(Exact sum, with possible carry-out limb)

Arbitrary-precision floating-point addition

Limbs are full 64-bit words

$a = 2^e \cdot$ |xxxxxxxx|xxxxxxxx|

$b = 2^f \cdot$ |xxxxxxxx|xxxxxxxx|xxxxxxxx|

The term with smaller exponent is right-shifted by $|f - e|$ bits, aligning the limb boundaries (several case distinctions)

```
                  |xxxxxxxx|xxxxxxxx|xxxxxxxx|
+                           |0000xxxx|xxxxxxxx|xxxx0000|

=  |0000000x|xxxxxxxx|xxxxxxxx|xxxxxxxx|xxxx0000|
```

(Exact sum, with possible carry-out limb)

```
=              |1xxxxxxx|xxxxxxxx|xxxxx000|    + ulp error
```

(Rounded and normalized)

Addition for fast dot product

Terms in dot product:
$t_k = (-1)^{s_k} 2^{e_k} \cdot$ |xxxxxxxx|xxxxxxxx|

Accumulator:
$s = 2^f \cdot$ |xxxxxxxxx|xxxxxxxxx|xxxxxxxxx|
$f = \max_k(e_k) + \text{bitlength}(\#\text{nonzero terms}) + 1$
Subtraction uses two's complement

Addition for fast dot product

Terms in dot product:
$t_k = (-1)^{s_k} 2^{e_k} \cdot$ |xxxxxxxx|xxxxxxxx|

Accumulator:
$s = 2^f \cdot$ |xxxxxxxx|xxxxxxxx|xxxxxxxx|
$f = \max_k(e_k) + \text{bitlength}(\#\text{nonzero terms}) + 1$
Subtraction uses two's complement

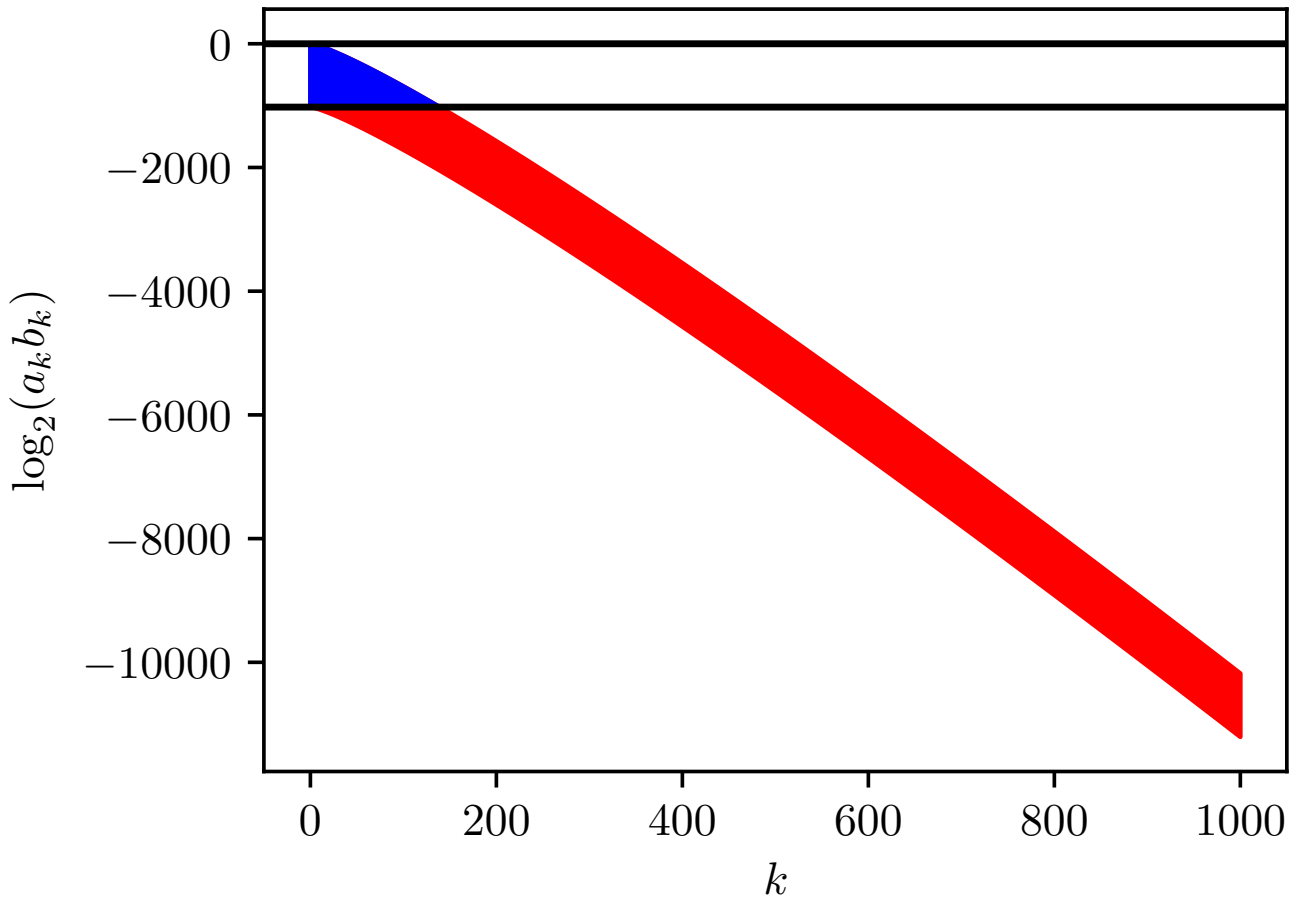
```
                |xxxxxxxx|xxxxxxxx|xxxxxxxx|
+                        |0000xxxx|xxxxxxxx|xxxx0000|
=               |xxxxxxxx|xxxxxxxx|xxxxxxxx|
                                              + ulp error
```

Multiplications in the dot product

- Using GMP's `mpn_mul` in general
- C code with inline ASM for $\leq 2 \times 2$ limb product, ≤ 3 limb accumulator
- Mulder's mulhigh (via MPFR) for 25 to 10000 limbs
- Karatsuba-type formula (3 instead of 4 real multiplications) for complex numbers at ≥ 128 limbs
- Optimal complexity for non-uniform input: insignificant limbs in the input vectors are discarded

Discarding insignificant limbs

Example: $\sum_{k=0}^{1000} a_k b_k,\ a_k = \dfrac{1}{k!},\ b_k = \dfrac{1}{\pi^k},\ p = 1024$

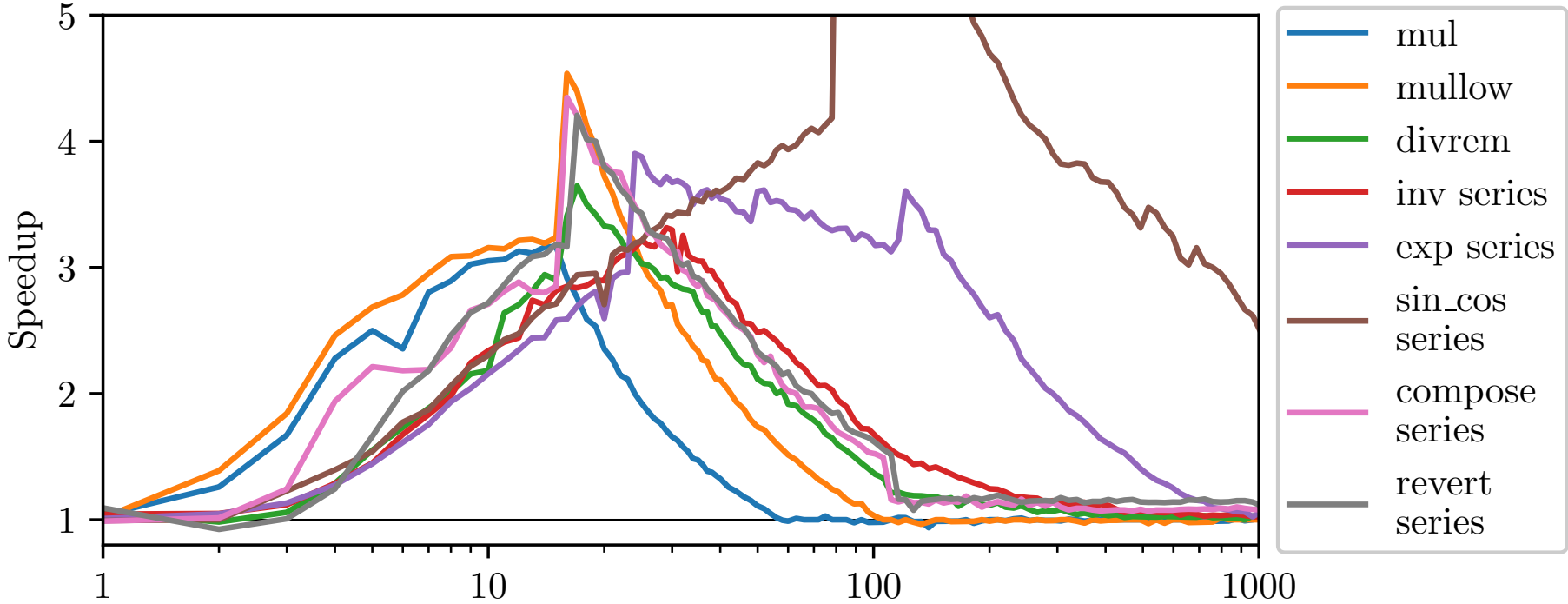


`arb_addmul`: 0.33 ms `arb_dot`: 0.035 ms

Speedup due to dot product: polynomial arithmetic

Speedup between Arb 2.14 and 2.15 due to switching from `arb_addmul`/`acb_addmul` to `arb_dot`/`acb_dot`

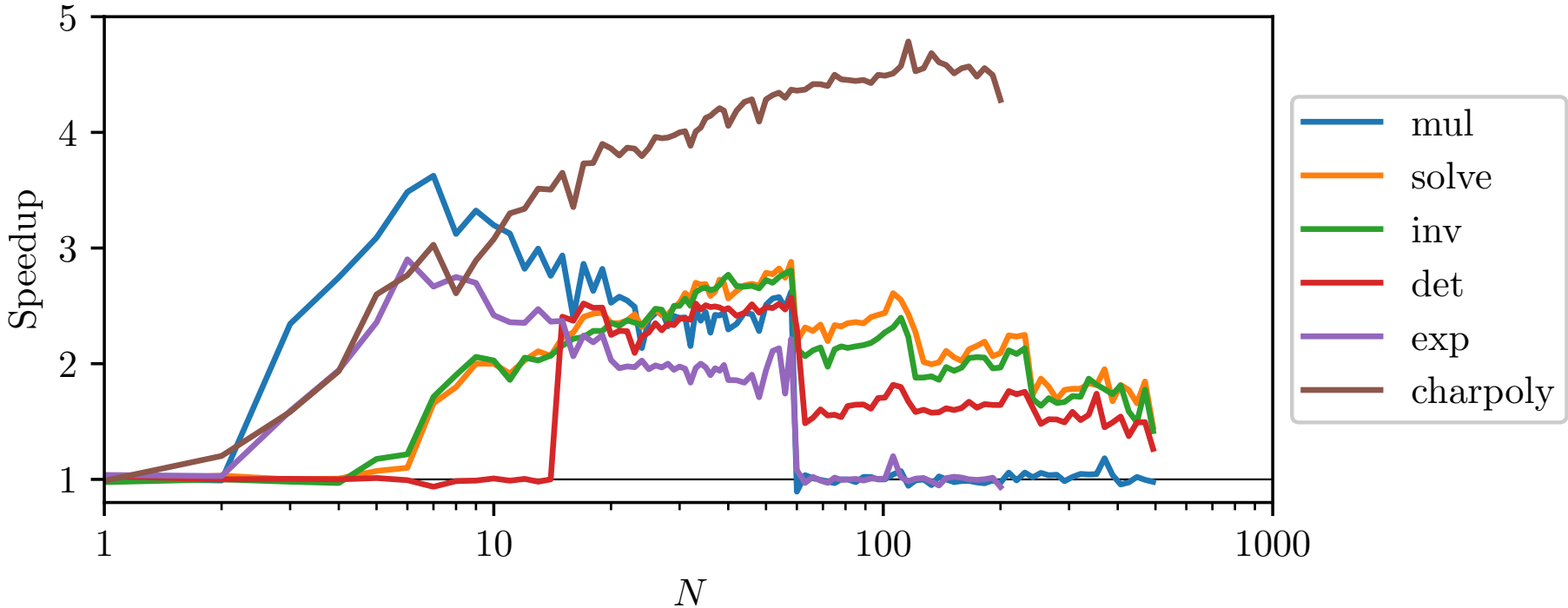
Complex polynomials (`acb_poly`), 64-bit precision



Speedup due to dot product: matrix arithmetic

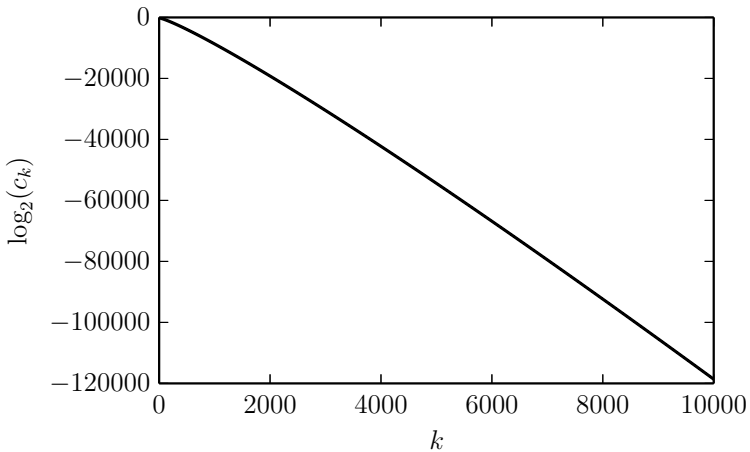
Speedup between Arb 2.14 and 2.15 due to switching from `arb_addmul`/`acb_addmul` to `arb_dot`/`acb_dot`

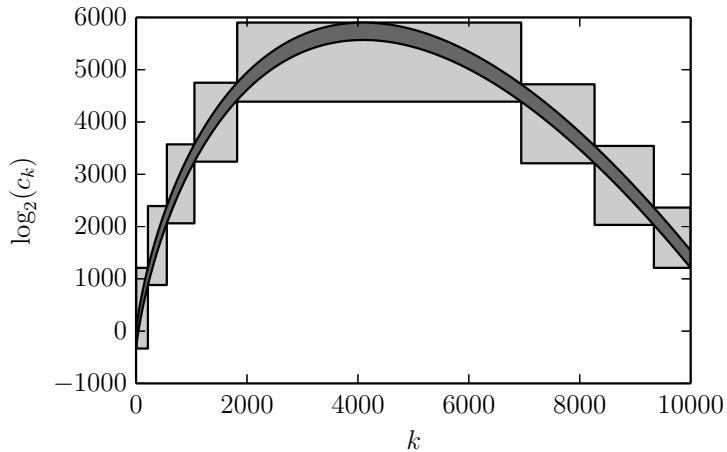
Complex matrices (`acb_mat`), 64-bit precision



BIG polynomial multiplication in Arb

- $(A+a)(B+b)$ via three multiplications AB, $|A|b$, $a(|B|+b)$
- Scaling $x \to 2^e x$, splitting into blocks





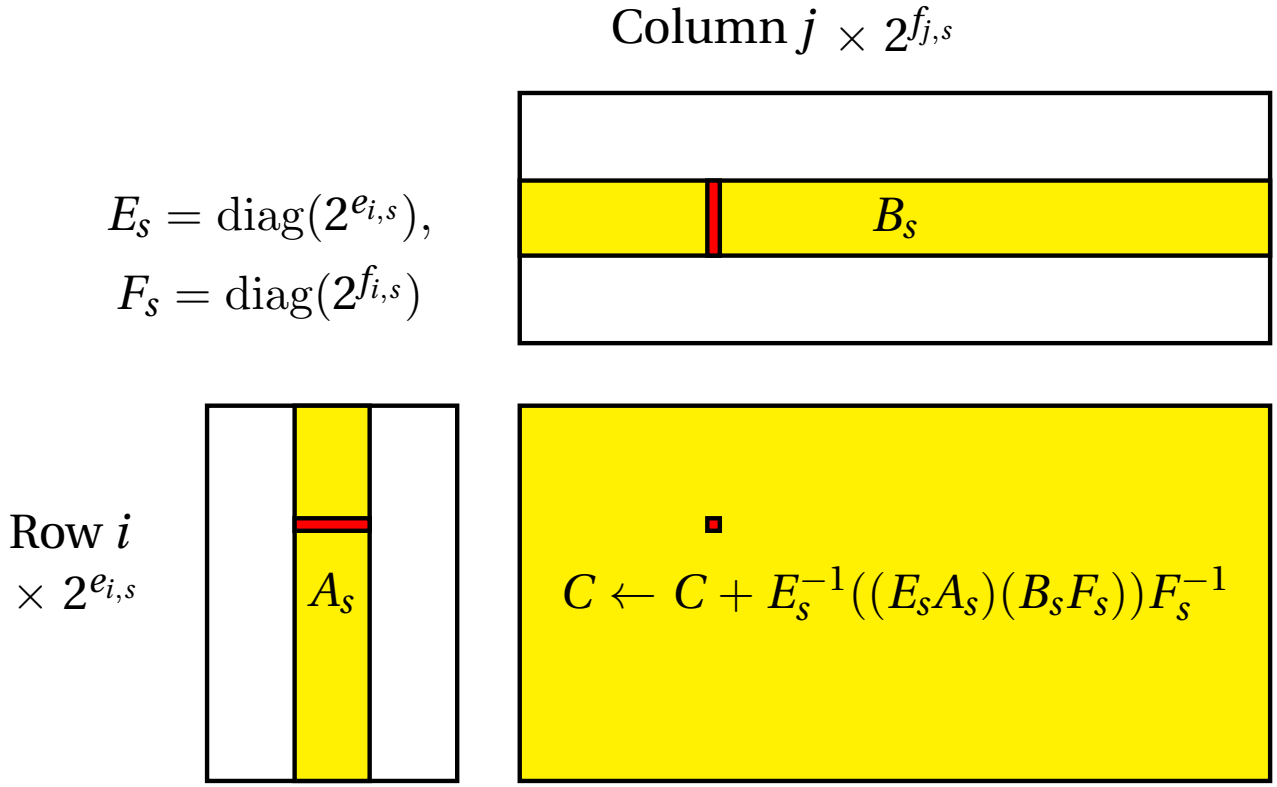
Transformation used to square the length-10 000 Taylor polynomial for $\exp(x) = \sum x^k/k!$ *at* 333 *bits precision*

- Blocks multiplied exactly over $\mathbb{Z}$ using FLINT (Karatsuba, Kronecker, Schönhage-Strassen FFT)
- For blocks up to length 1000 in $|A|b$, $a(|B|+b)$, use `double`

[J., IEEE Trans. Comp., 2017], following [van der Hoeven, 2008]

BIG matrix multiplication in Arb (new in Arb 2.14)

- $(A+a)(B+b)$ via three multiplications AB, $|A|b$, $a(|B|+b)$
- Scaling + splitting into blocks



- Blocks multiplied exactly over $\mathbb{Z}$ using FLINT (Strassen, small primes multimodular multiplication)
- For blocks in $|A|b$, $a(|B|+b)$, use `double`

Timings: matrix multiplication

	Uniform				Pascal	
p	QD	MPFR	Arb (dot)	Arb (block)	Arb (dot)	Arb (block)
			$N = 300$			
53		0.96	0.51	0.13	0.37	0.57 (3)
106	0.30	1.2	0.69	0.23	0.47	0.70 (3)
212	3.0	3.0	2.2	0.34	1.8	1.2 (3)
848		7.9	6.2	1.2	5.1	2.4 (2)
3392		46	47	6.0	44	7.3
			$N = 1000$			
53		36	19	3.6	12	20 (10)
106	11	44	25	5.6	14	23 (10)
212	111	110	76	8.2	43	35 (9)
848		293	258	27	122	80 (5)
3392		1725	1785	115	1280	226 (2)

(#) is the number of blocks $A_s B_s$ used by the block algorithm

Numerically stable ball linear algebra

Gaussian elimination (GE) in ball/interval arithmetic is unstable in general, even for well-conditioned matrices

Numerically stable ball linear algebra

Gaussian elimination (GE) in ball/interval arithmetic is unstable in general, even for well-conditioned matrices

Solving $AX = B$ (Hansen-Smith, 1967):

- Compute $R \approx A^{-1}$ using (nonrigorous) floating-point GE, then solve $(RA)X = RB$
- GE in ball arithmetic applied to $RA \approx I$ is stable (can also use perturbation norm bounds)

Numerically stable ball linear algebra

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Computing $\det(A)$ (Rump, 2010):

- Compute approximate LU factorization $A \approx PLU$ and approximate inverses $L' \approx L^{-1}$, $U' \approx U^{-1}$, then construct $B = L'P^{-1}AU'$ using ball arithmetic
- The determinant of $B \approx I$ can be enclosed using Gershgorin circles (warning: correctness not obvious!)

Several methods

- `arb_mat_solve_lu` – GE in ball arithmetic; unstable
- `arb_mat_solve_precond` – Hansen-Smith algorithm; 4-5 times slower, stable
- `arb_mat_solve` – automatic choice; GE in ball arithmetic if $N \leq 4$ or $p > 10N$, otherwise Hansen-Smith
- `arb_mat_approx_solve` – GE in floating-point arithmetic (no error bounds); stable; uses `arb_approx_dot` in the basecase

Block recursive methods used everywhere to reduce everything to matrix multiplication asymptotically

Timings: arbitrary-precision linear solving

N	p	Eigen	Julia	Arb (approx)	Arb (ball)
10	53	0.00028	0.000066	0.000021	0.00013*
10	106	0.00029	0.000070	0.000025	0.000040
10	212	0.00033	0.00010	0.000055	0.000074
10	848	0.00043	0.00022	0.00014	0.00016
10	3392	0.0012	0.0010	0.00088	0.00090
100	53	0.051	0.064	0.0069	0.040*
100	106	0.054	0.070	0.0084	0.049*
100	212	0.080	0.10	0.024	0.10*
100	848	0.16	0.22	0.080	0.35*
100	3392	0.71	0.90	0.49	0.50
1000	53	37	301	2.3	13*
1000	106	39	401	3.3	20*
1000	212	64	488	6.6	36*
1000	848	132	947	24	118*
1000	3392	601	2721	153	609*

* The Hansen-Smith algorithm is used

Timings: eigendecomposition

Time to diagonalize a size-N nonsymmetric complex matrix

N	p	Julia	Arb (approx)	Arb (Rump)	Arb (vdHM)
10	128	0.021	0.0036	0.0082	0.0045
10	384	0.043	0.011	0.022	0.013
100	128	8.8	2.5	18.2	2.9
100	384	18.5	8.7	59	9.8
1000	128	$> 3 \cdot 10^4$	2764		2981
1000	384		9358		9877

- approx – numerical QR algorithm (no error bounds)
- QR + $O(N^4)$ Rump certification algorithm
- QR + $O(N^3)$ van der Hoeven - Mourrain certification

Summary: faster arithmetic and linear algebra

Improvements from Arb 2.13 to Arb 2.16:

- $2 - 5\times$ speedup (at low precision) for basecase polynomial and matrix arithmetic, using fast dot product
- Order-of-magnitude speedup for multiplying big matrices (without loss of accuracy), using scaled blocks
- Numerically stable linear algebra + taking advantage of matrix multiplication
- Bonus: certified arbitrary-precision eigendecomposition

Still a work in progress!