Fungrim: The Mathematical Functions Grimoire

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What is Fungrim?

`http://fungrim.org`

An attempt to make a better (at least for me)

1. reference work
2. software library for symbolic computation

for special functions

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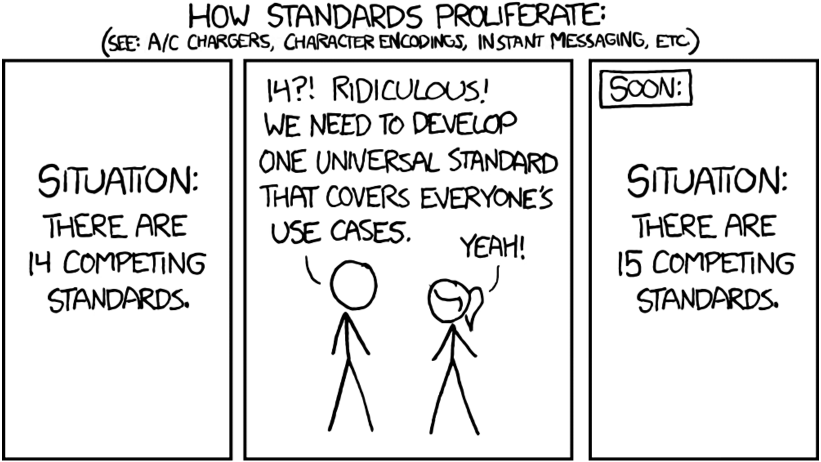
grimoire = book of magic formulas

Relevant XKCD



`https://xkcd.com/927/`

Problems with existing reference works

Digital Library of Mathematical Functions

THE MATHEMATICAL FUNCTIONS SITE

WIKIPEDIA
The Free Encyclopedia

	`dlmf.nist.gov`	`functions.wolfram.com`	`wikipedia.org`
Open source?	×	×	✓
Symbolic?	×	✓	×
Good	Good presentation (edited by experts)	Well-structured, huge	Usually comprehensive
Bad	Terse, missing useful formulas, sometimes vague	*Mathematica* quirks and bugs, sometimes ugly formulas, missing some categories of info	Text-heavy, much trivia, often vague, inconsistent

Some reasons why the literature is frustrating to use

1. Vague or missing definitions
2. Conditions on variables not stated, ambiguous, or depend on non-local context
3. Implicit special cases, limits, analytic continuation, ...
4. The wanted formula can be derived by combining equation (43) with theorems 5 and 12... in a simple 10-page calculation, left as an exercise for the reader
5. The dreaded "$\approx$" sign
6. Errors (typos or more serious)
7. Text text text text text text text text text text

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I'm personally as guilty as anyone, on all counts

Content goals for Fungrim

- Formulas as symbolic, machine-readable theorems
- Symbols have a globally consistent meaning
- Explicit assumptions for all free variables

$$|T_n(x)| \le 1, \quad n \in \mathbb{Z} \text{ and } x \in [-1, 1]$$

```
Entry(ID("15dd69"),
    Formula(LessEqual(Abs(ChebyshevT(n, x)), 1)),
    Variables(n, x),
    Assumptions(And(Element(n, ZZ),
        Element(x, ClosedInterval(-1, 1)))))
```

- Scope not limited by paper edition constraints
- Good coverage of inequalities, with explicit constants

Presentation goals for Fungrim

- Simple and fast to browse (including mobile!)
- Permanent ID and URL for each formula
- Beautiful formula rendering (symbolic expressions $\to$ TeX $\to$ KaTeX $\to$ HTML)
- Instant access to TeX code to copy and paste
- Instant access to symbolic representation
- Links to symbol definitions
- TODO: export to other languages, search functionality, browsing based on metadata

Non-goals (for now)

Formal proofs

- Randomized testing (to be done!) should be adequate to provide a high level of reliability

Fully computer-generated content

- Related: the Dynamic Dictionary of Mathematical Functions (`http://ddmf.msr-inria.inria.fr/1.9.1/ddmf`)

Covering all of mathematics

- Just special functions and elements of classical analysis and number theory

Long-term goal: symbolic computation

Applications of a library of formulas

- Automatic (or manual) term rewriting
- Code generation, testing mathematical software

What's missing in existing projects?

- Not open source
- More narrow scope
- Mathematical knowledge encoded implicitly in text or code (and mixed with implementation details), not as symbolically readable data
- Missing or inconsistent assumptions

Inspiration 1: Rubi by Albert D. Rich

`https://rulebasedintegration.org`

> *[Rubi] uses pattern matching to uniquely determine which of its over 6600 integration rules to apply to a given integrand*

> *Rubi dramatically out-performs other symbolic integrators, including Maple and Mathematica*

> *Certainly much of analysis including equation solving, expression simplification, differentiation, summation, limits, etc. can be automated using this paradigm*

Inspiration 2: current computer algebra systems (and how broken they are)

- Too zealous “simplification”
- Mathematical inconsistencies or bugs

A simple symbolic integral: $\int_1^2 x^a dx$

Mathematica:

```
In[5]:= Integrate[x ^ (-1) , {x, 1, 2}]

Out[5]= Log[2]


In[7]:= Integrate[x ^ a, {x, 1, 2}]
```

Out[7]= $\frac{-1+2^{1+a}}{1+a}$

```
In[8]:= Integrate[x ^ a, {x, 1, 2}] /. (a → -1)
```

··· Power: Infinite expression $\frac{1}{0}$ encountered.

··· Infinity : Indeterminate expression 0 ComplexInfinity encountered.

```
Out[8]= Indeterminate
```

A simple symbolic integral: $\int_1^2 x^a dx$

SymPy does the right thing:

```
>>> integrate(x**a, (x, 1, 2))
Piecewise((2**(a + 1)/(a + 1) - 1/(a + 1),
    (a > -oo) & (a < oo) & Ne(a, -1)), (log(2), True))

>>> integrate(x**a, (x, 1, 2)).subs(a, -1)
log(2)
```

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>>> integrate(x**a, (x, 1, 2)).subs(a, -1)
log(2)
```

Well, almost:

```
>>> integrate(x**a, (x, 1, 2)).subs(a, I)
Traceback (most recent call last):
  ...
TypeError: Invalid comparison of complex I
```

What is ${}_1F_1(-1, -1, 1)$?

Mathematica:

```
In[•]:= Hypergeometric1F1[n, m, 1] /. {m → -1, n → -1, x → 1}
Out[•]= 2

In[•]:= (Hypergeometric1F1[n, m, x] /. {m → n}) /. {n → -1, x → 1}
Out[•]= e
```

SymPy:

```
>>> simplify(hyper([n],[m],x).subs({m:-1, n:-1, x:1}))
2
>>> simplify(hyper([n],[m],x).subs(m, n)).subs({n:-1, x:1})
E
```

What is ${}_1F_1(-1, -1, 1)$?

http://fungrim.org/entry/dec042/

$${}_1F_1(-n, b, z) = \sum_{k=0}^{n} \frac{(-n)_k}{(b)_k} \frac{z^k}{k!}$$

Assumptions:
$n \in \mathbb{Z}_{\ge 0}$ and $b \in \mathbb{C}$ and not $(b \in \{0, -1, \ldots\}$ and $b > -n)$ and $z \in \mathbb{C}$

http://fungrim.org/entry/be533c/

$${}_1F_1(a, b, z) = e^z \, {}_1F_1(b - a, b, -z)$$

Assumptions: $a \in \mathbb{C}$ and $b \in \mathbb{C} \setminus \{0, -1, \ldots\}$ and $z \in \mathbb{C}$

Example: derivative of the modular $\lambda(\tau)$ function

```
In[91]:= N[D[ModularLambda[tau], tau] /. tau → 1/2 + 3 I, 10]
         ND[ModularLambda[tau], tau, 1/2+ 3 I, Scale → 10 ^ -16,
          WorkingPrecision→30]

Out[91]= -0.004056396698+5.237591×10^-6 i

Out[92]= -0.004056396698+5.237591×10^-6 i

In[93]:= N[D[ModularLambda[tau], tau] /. tau → 1/2 + I/3, 10]
         ND[ModularLambda[tau], tau, 1/2+ I/3, Scale → 10 ^ -16,
          WorkingPrecision→ 30]

Out[93]= 0.332090725+4.583245973 i

Out[94]= -0.74720413-10.31230344 i
```

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Mathematica uses an elliptic integral to express $\lambda'(\tau)$. This is not valid everywhere because of branch cuts!

Fast and simple versus correct

R. Corless and D. Jeffrey, “Well… It Isn’t Quite That Simple”, ACM SIGSAM Bulletin, 1992:

> *The automatic exploration of conditions or alternative results requires considerable computational resources, and for the sake of speed there is an attraction to picking one ’obvious’ answer. [...] The difficulty is to balance efficiency against correctness.*

27 years later, what is the right balance?