Efficient implementation of the Hardy-Ramanujan-Rademacher formula

or: Partitions in the quintillions

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The partition function

$p(n)$ counts the number of ways n can be written as the sum of positive integers without regard to order.

Example: $p(4) = 5$ since

$$(4) = (3+1) = (2+2) = (2+1+1) = (1+1+1+1)$$

$(p(n))_{n=0}^{\infty} = 1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42 \ldots$

Growth of $p(n)$

$p(10) = 42$
$p(100) = 190569292$
$p(1000) = 24061467864032622473692149727991 \approx 2.4 \times 10^{31}$
$p(10000) \approx 3.6 \times 10^{106}$
$p(100000) \approx 2.7 \times 10^{346}$
$p(1000000) \approx 1.5 \times 10^{1107}$

$$p(n) \sim \frac{1}{4n\sqrt{3}} \; e^{\pi\sqrt{2n/3}}$$

$p(n)$ has $\sim n^{1/2}$ digits

Euler's method to compute $p(n)$

Generating function (Euler, 1748):

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k} = \left(\sum_{k=-\infty}^{\infty} (-1)^k x^{k(3k-1)/2} \right)^{-1}$$

Recursive formula:

$$p(n) = \sum_{k=1}^{n} (-1)^{k+1} \left(p\left(n - \frac{k(3k-1)}{2}\right) + p\left(n - \frac{k(3k+1)}{2}\right) \right)$$

Complexity: $O(n^{3/2})$ integer operations, $O(n^2)$ bit operations

Asymptotically fast vector computation

Use fast power series arithmetic to expand

$$\frac{1}{f(x)} = p(0) + p(1)x + \ldots + p(n)x^n + O(x^{n+1})$$

The complexity is **quasi-optimal** for computing $p(0), \ldots, p(n)$ **simultaneously**:

- $O(n^{3/2+o(1)})$ bit operations over $\mathbb{Z}$
- $O(n^{1+o(1)})$ bit operations over $\mathbb{Z}/m\mathbb{Z}$ for fixed m

Calkin et al (2007): computation of $p(n)$ mod m for all $n \leq 10^9$ and primes $m \leq 103$

The Hardy-Ramanujan-Rademacher formula

There is a better way to compute an **isolated** value of $p(n)$, due to Hardy and Ramanujan (1917), Rademacher (1936):

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \sqrt{k}\, A_k(n) \frac{d}{dn} \left(\frac{1}{\sqrt{n - \frac{1}{24}}} \sinh \left[\frac{\pi}{k} \sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)} \right] \right)$$

$$A_k(n) = \sum_{\substack{0 \le h < k \\ \gcd(h,k)=1}} e^{\pi i \left[s(h,k) - \frac{1}{k} 2nh \right]}$$

$$s(h,k) = \sum_{i=1}^{k-1} \frac{i}{k} \left(\frac{hi}{k} - \left\lfloor \frac{hi}{k} \right\rfloor - \frac{1}{2} \right)$$

Explicit error bound by Rademacher: can truncate after $O(n^{1/2})$ terms such that the error is smaller than $1/2$

How fast can we compute $p(n)$ using the HRR formula?

1938: Lehmer manually computes $p(599)$, $p(721)$

1995: Odlyzko claims that $p(n)$ can be computed in **quasi-optimal time**, but does not give a proof or an algorithm.

A few years ago:

- Implementations in several computer algebra systems: Pari/GP, Maple, Mathematica, Sage, etc. There are large **differences in performance**. Many versions give **wrong values**.
- No algorithmic analysis or implementation studies in the literature
- Largest reported values: $p(n), n \approx 10^9$

New study

F. J. (2012). "Efficient implementation of the Hardy–Ramanujan–Rademacher formula."
LMS J. Comp. Math. 15(1): 341-359.

- Proof that $p(n)$ can be computed in quasi-optimal time
- A new implementation, running up to $\sim$ 500 times faster than previous software (open source, part of FLINT, `http://flintlib.org`)
- Error bounds for the main numerical parts of the algorithm
- Discussion of implementation issues and practical optimizations
- Large-scale $p(n)$ computation, including generation of congruences

Quasi-optimality for isolated values of $p(n)$

Theorem

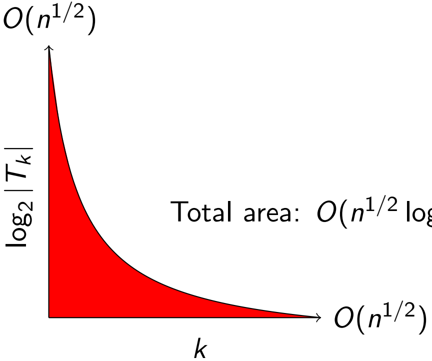
$p(n)$ can be computed using $O(n^{1/2} \log^{4+o(1)} n) = O(n^{1/2+o(1)})$ bit operations.

This is quasi-optimal since $p(n)$ has $\Theta(n^{1/2})$ bits.

- Unlike many sequences for which quasi-optimal algorithms are known, $p(n)$ is not P-finite (holonomic)
- Quasi-optimal algorithms are not known for e.g. isolated Bell numbers (set partitions)

Cost of numerical evaluation

$p(n) = \sum_{k=0}^{N} T_k + \varepsilon \qquad N = O(n^{1/2}), \quad \log_2 |T_k| = O(n^{1/2}/k)$



We can compute $p(n)$ in quasi-optimal time, if we can approximate T_k in quasi-optimal time.

Numerical evaluation of elementary functions

$$T_k = (A_k(n) : \text{sum of roots of unity}) \times (\text{hyperbolic function})$$

All numerical evaluation can be reduced to **elementary functions**:

- exp
- log
- sin
- sinh
- ...

Elementary functions can be evaluated to b-bit accuracy in quasi-optimal time $O(b^{1+o(1)})$.

Evaluating exponential sums

$$A_k(n) = \sum_{\substack{0 \le h < k \\ \gcd(h,k)=1}} e^{\pi i \left[s(h,k) - \frac{1}{k} 2nh\right]}$$

$$s(h,k) = \sum_{i=1}^{k-1} \frac{i}{k} \left(\frac{hi}{k} - \left\lfloor \frac{hi}{k} \right\rfloor - \frac{1}{2} \right)$$

Naively:

- $O(k^2)$ (integer/elementary function) operations for $A_k(n)$
- $O(n^{3/2})$ total (integer/elementary function) operations for $p(n)$

We need to get the cost for $A_k(n)$ down to $O(\log^c k)$ (integer/elementary function) operations!

Fast computation of Dedekind sums

Let $0 < h < k$ and let $k = r_0, r_1, \ldots, r_{m+1} = 1$ be the sequence of remainders in the Euclidean algorithm for $\gcd(h, k)$. Then

$$s(h,k) = \frac{(-1)^{m+1} - 1}{8} + \frac{1}{12} \sum_{j=1}^{m+1} (-1)^{j+1} \frac{r_j^2 + r_{j-1}^2 + 1}{r_j r_{j-1}}.$$

Fraction-free version by Knuth (1975).

- $O(\log k)$ integer or rational operations to evaluate $s(h,k)$
- $O(k \log k)$ integer operations to evaluate $A_k(n)$
- $O(n \log n)$ integer operations to evaluate $p(n)$

Still not good enough!

Evaluating $A_k(n)$ using prime factorization

Whiteman (1956):

- If $k = p^e$, then

$$A_k(n) = \sqrt{\frac{s}{t}} \cos\left(\frac{\pi r}{24k}\right)$$

- If $k = k_1 k_2$, $\gcd(k_1, k_2) = 1$, then

$$A_k(n) = A_{k_1}(n_1) A_{k_2}(n_2)$$

$r, s, t, n_1, n_2 \in \mathbb{Z}$ are determined by equations involving modular square roots, GCDs, Jacobi symbols, case distinctions.

Algorithm: factor k into prime powers to write $A_k(n)$ as a product of $O(\log k)$ cosines. **Now the numerical evaluation becomes fast enough!**

Cost of integer arithmetic

Factoring: we do not know how to factor k in $O(\log^c k)$ time. However, we can factor $1, \ldots, n^{1/2}$ simultaneously in time $O(n^{1/2} \log n)$.

Integer arithmetic: multiplication, GCD, ...: $O(\log^{1+o(1)} k)$

Square roots mod p:

- $O(\log^{3+o(1)} p)$ using the Shanks-Tonelli algorithm
- $O(\log^{2+o(1)} p)$ using Cipolla's algorithm
- Must know a quadratic nonresidue mod p (by a result of Erdős, a table for all $p < n^{1/2}$ can be precomputed sufficiently quickly)

Total cost of integer operations for $A_k(n)$: $O(\log^{3+o(1)} k)$

New implementation

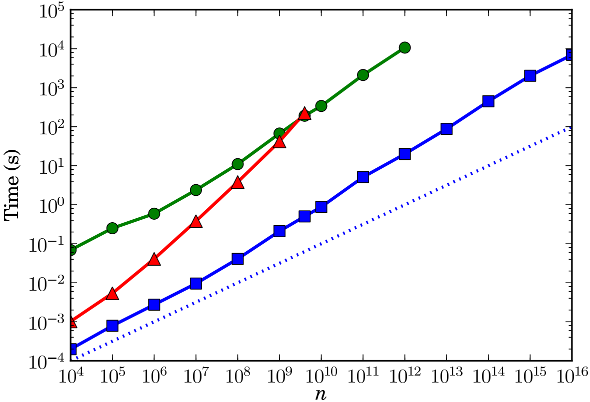
2011:

- Using FLINT (integers) + MPFR (arbitrary-precision floats)
- *A priori* floating-point error bounds for the body of the algorithm
- Many numerical "tricks" **without complete error bounds**
  - Fast algorithms for π, roots of unity, ...
  - Using hardware double-precision arithmetic

2013:

- Using FLINT + MPFR + Arb (new ball arithmetic library)
- "Tricks" reimplemented as proper Arb library functions, with proofs
- Code for $p(n)$ is simpler, **with complete error bounds**

Timings for $p(n)$ (2011)



Mathematica 7 (green circles)

Sage 4.7 (red triangles)

FLINT (blue squares)

Timings for $p(n)$ (2011)

n	Mathematica 7	Sage 4.7	FLINT	First term
10^4	69 ms	1 ms	0.20 ms	
10^5	250 ms	5.4 ms	0.80 ms	
10^6	590 ms	41 ms	2.74 ms	
10^7	2.4 s	0.38 s	0.010 s	
10^8	11 s	3.8 s	0.041 s	
10^9	67 s	42 s	0.21 s	43%
10^{10}	340 s		0.88 s	53%
10^{11}	2,116 s		5.1 s	48%
10^{12}	10,660 s		20 s	49%
10^{13}			88 s	48%
10^{14}			448 s	47%
10^{15}			2,024 s	39%
10^{16}			6,941 s	45%
10^{17}			27,196* s	33%
10^{18}			87,223* s	38%
10^{19}			350,172* s	39%

Large values of $p(n)$

n	Decimal expansion	Num. digits	Terms	Error
10^{12}	6129000962 . . . 6867626906	1,113,996	264,526	10^{-7}
10^{13}	5714414687 . . . 4630811575	3,522,791	787,010	10^{-8}
10^{14}	2750960597 . . . 5564896497	11,140,072	2,350,465	10^{-8}
10^{15}	1365537729 . . . 3764670692	35,228,031	7,043,140	10^{-9}
10^{16}	9129131390 . . . 3100706231	111,400,846	21,166,305	10^{-9}
10^{17}	8291300791 . . . 3197824756	352,280,442	63,775,038	10^{-9}
10^{18}	1478700310 . . . 1701612189	1,114,008,610	192,605,341	10^{-10}
10^{19}	5646928403 . . . 3674631046	3,522,804,578	582,909,398	10^{-11}

The number of partitions of ten quintillion:
$p(10^{19}) = p(10000000000000000000) \approx 5.65 \times 10^{3,522,804,577}$

3.5 GB output, 97 CPU hours, $\sim$ 150 GB memory

New timings (2013, on slightly faster hardware)

n	Mathematica 8.0	FLINT*	Arb**
10^6	0.328 s	**0.00147 s**	0.00478 s
10^9	23.7 s	**0.142 s**	0.181 s
10^{12}	2458 s	**11.32 s**	11.50 s
10^{15}	307810 s	1109 s	**1097 s**
10^{18}		66738 s	**57102 s**

* 2011 implementation: using MPFR + hardware doubles (with incomplete error bounds)
** 2013 implementation: using ball arithmetic throughout to provably determine $p(n)$

Partition function congruences

Ramanujan (1919): for all $k \in \mathbb{N}$,

$$p(5k+4) \equiv 0 \pmod 5$$
$$p(7k+5) \equiv 0 \pmod 7$$
$$p(11k+6) \equiv 0 \pmod{11}$$

Ono (2000): for every prime $m \geq 5$, there exist infinitely many congruences of the type

$$p(Ak+B) \equiv 0 \bmod m$$

Algorithm to generate congruences (Weaver, 2001)

Defining tuple: (m, ℓ, ε)

- $m \in \{13, 17, 19, 23, 29, 31\}$
- $\ell \geq 5$ prime
- $\varepsilon \in \{-1, 0, 1\}$

For certain X, Y, Z where $X = O(\ell^2)$, check **the single case**

$$p(X) \equiv Y \bmod Z$$

If true, we obtain explicit A, B of size $O(\ell^4)$ such that **for all** k,

$$p(Ak + B) \equiv 0 \bmod m$$

For a given tuple (m, ℓ, ε), there are $O(\ell)$ such pairs A, B, enumerated by an additional parameter δ.

Weaver's table

Weaver gives 76,065 congruences (167 tuples), obtained from a table of all $p(n)$ with $n < 7.5 \times 10^6$ (computed using the recursive Euler algorithm).

Limit on $\ell \approx 10^3$

Example: $m = 31$

$\varepsilon = 0$: $\ell = 107, 229, 283, 383, 463$

$\varepsilon \neq 0$: $(\ell, \varepsilon) = (101, 1), (179, 1), (181, 1), (193, 1), (239, 1), (271, 1)$

New table

Testing all $\ell < 10^6$ resulted in 22 billion new congruences (70,359 tuples).

This involved evaluating $p(n)$ for $6(\pi(10^6) - 3) = 470,970$ distinct n, in parallel on ≈ 40 cores (hardware at University of Warwick, courtesy of Bill Hart)

m	$\varepsilon = 0$	$\varepsilon = +1$	$\varepsilon = -1$	Congruences	CPU	Max n
13	6,189	6,000	6,132	5,857,728,831	448 h	5.9×10^{12}
17	4,611	4,611	4,615	4,443,031,844	391 h	4.9×10^{12}
19	4,114	4,153	4,152	3,966,125,921	370 h	3.9×10^{12}
23	3,354	3,342	3,461	3,241,703,585	125 h	9.5×10^{11}
29	2,680	2,777	2,734	2,629,279,740	1,155 h	2.2×10^{13}
31	2,428	2,484	2,522	2,336,738,093	972 h	2.1×10^{13}
All	23,376	23,367	23,616	22,474,608,014	3,461 h	

Examples of new congruences

Example 1: $(13, 3797, -1)$ with $\delta = 2588$ gives

$$p(711647853449k + 485138482133) \equiv 0 \bmod 13$$

which we may easily confirm for $k \leq 100$ by evaluation.

Example 2: $(29, 999959, 0)$ with $\delta = 999958$ gives

$$p(28995244292486005245947069k + 28995221336976431135321047)$$

$$\equiv 0 \bmod 29$$

This is out of reach for explicit evaluation ($n \approx 10^{25}$)

Download the data

http://www.risc.jku.at/people/fjohanss/partitions/

or

http://sage.math.washington.edu/home/fredrik/partitions/

Comparison of algorithms for vector computation

n	Series ($\mathbb{Z}/13\mathbb{Z}$)	Series ($\mathbb{Z}$)	HRR (all)	HRR (sparse)
10^4	0.01 s	0.1 s	1.4 s	0.001 s
10^5	0.13 s	4.1 s	41 s	0.008 s
10^6	1.4 s	183 s	1430 s	0.08 s
10^7	14 s			0.7 s
10^8	173 s			8 s
10^9	2507 s			85 s

HRR competitive over $\mathbb{Z}$: when n/c values are needed (our improvement: $c \approx 10$ vs $c \approx 1000$)

HRR competitive over $\mathbb{Z}/m\mathbb{Z}$: when $O(n^{1/2})$ values are needed (speedup for Weaver's algorithm: 1-2 orders of magnitude).

Most important advantages: little memory, parallel, resumable

Conclusions

- Isolated values of $p(n)$ can be computed fast, both in theory and in practice
- The HRR formula allows performing computations that are impractical with power series methods
- Care is required for both asymptotics and implementation details
- Generalizations: other HRR-type series for special types of partitions (into distinct parts, etc), and possibly other number-theoretical computations