# Making change for $10^{20}$

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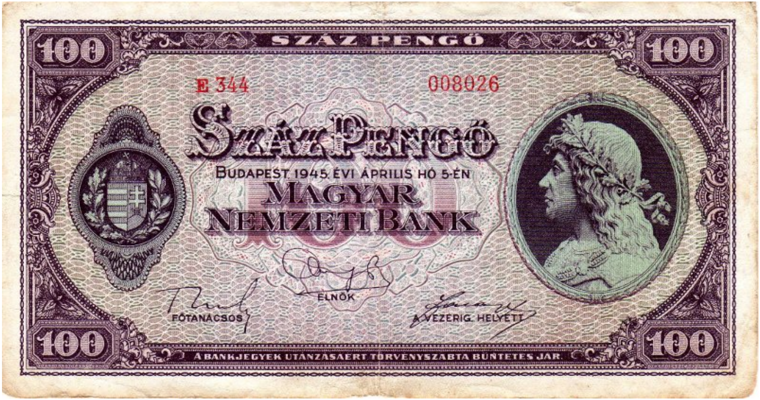
RISC Algorithmic Combinatorics Seminar

# Hungarian pengő (1 P, 1926)

# $10^2$ pengő (April 1945)




100 pengő

# $10^3$ pengő (July 1945)


1000 pengő

# $10^4$ pengő (July 1945)


10,000 pengő

# $10^5$ pengő (October 1945)


100,000 pengő

# $10^6$ pengő (November 1945)


1,000,000 pengő

# $10^7$ pengő (November 1945)


10,000,000 pengő

# $10^8$ pengő (March 1946)


100,000,000 pengő

# $10^9$ pengő (March 1946)


1,000,000,000 pengő

# $10^{10}$ pengő (April 1946)


10,000 milpengő = 10,000,000,000 pengő

# $10^{11}$ pengő (April 1946)


100,000 milpengő = 100,000,000,000 pengő

# $10^{12}$ pengő (May 1946)


1,000,000 milpengő = 1,000,000,000,000 pengő

# $10^{13}$ pengő (May 1946)


10,000,000 milpengő = 10,000,000,000,000 pengő

# $10^{14}$ pengő (June 1946)


100,000,000 milpengő = 100,000,000,000,000 pengő

# $10^{15}$ pengő (June 1946)


1,000,000,000 milpengő = 1,000,000,000,000,000 pengő

# $10^{16}$ pengő (June 1946)


10,000 b.pengő = 10,000,000,000,000,000 pengő

# $10^{17}$ pengő (June 1946)


100,000 b.pengő = 100,000,000,000,000,000 pengő

# $10^{18}$ pengő (June 1946)


1 million b.pengő = 1,000,000,000,000,000,000
(1 quintillion) pengő

# $10^{19}$ pengő (June 1946)


10 million b.pengő = 10,000,000,000,000,000,000
(10 quintillion) pengő

# $10^{20}$ pengő (June 1946)


100 million b.pengő = 100,000,000,000,000,000,000 (100 quintillion) pengő

# August 1946

# Making change (partitions)


$$10^{20} = 1 + 3 + 3 + 99999999999999999993$$

Number of partitions: $p(10^{20})$

# New result

**Theorem (FJ+Popeye, 2014)**

*There are exactly*

$$18381765083448 82\ldots 231756788091448$$

*(11,140,086,260 digits) different ways to make change for a* $10^{20}$*-pengő banknote using stacks of 1-pengő coins.*

# Growth of $p(n)$

$$p(n) \sim \frac{1}{4n\sqrt{3}}\ e^{\pi\sqrt{2n/3}}$$

$p(n)$ has $\sim n^{1/2}$ digits

$p(10) = 42$
$p(100) = 190569292$
$p(1000) \approx 2.4 \times 10^{31}$
$p(10000) \approx 3.6 \times 10^{106}$
$p(100000) \approx 2.7 \times 10^{346}$
$p(1000000) \approx 1.5 \times 10^{1107}$

# The HRR formula

$$p(n) = \sum_{k=1}^{\infty} \frac{\sqrt{k}\,A_k(n)}{\pi\sqrt{2}} \frac{d}{dn} \left( \frac{\sinh\left[\frac{\pi}{k}\sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)}\right]}{\sqrt{n - \frac{1}{24}}} \right)$$

$$A_k(n) = \sum_{\substack{0 \le h < k \\ \gcd(h,k)=1}} e^{\pi i \left[s(h,k) - \frac{1}{k} 2nh\right]}$$

$$s(h,k) = \sum_{i=1}^{k-1} \frac{i}{k} \left( \frac{hi}{k} - \left\lfloor \frac{hi}{k} \right\rfloor - \frac{1}{2} \right)$$

Hardy and Ramanujan (1917), Rademacher (1936)

# From a past seminar...

**Theorem (FJ, 2011)**
*$p(n)$ can be computed in time $O(n^{1/2} \log^{4+o(1)} n)$.*

Computations: $p(10^k)$ up to $k = 19$

22 billion congruences $p(Ak + B) \equiv 0 \bmod m$

# Improved implementation (Arb library)

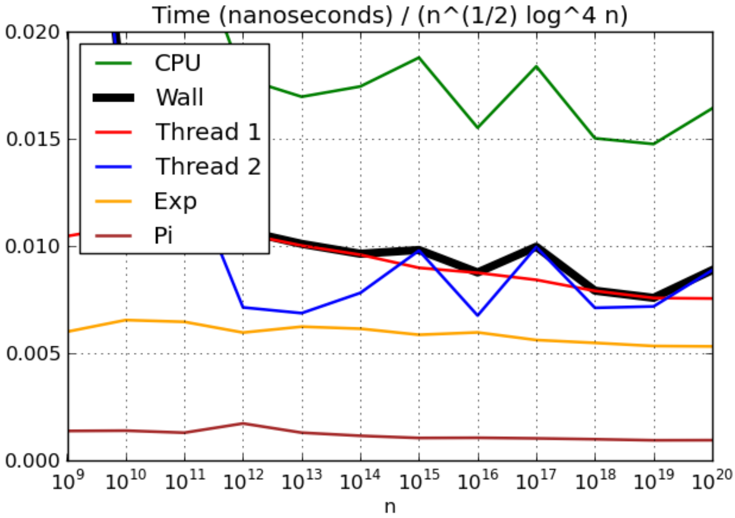
- Rigorous error bounds
- $3\times$ faster
- $3\times$ less memory

| Time | Memory | 2011 | 2014 |
|---|---|---|---|
| 10 hours | 15 GB | $p(10^{17})$ | $p(10^{18})$ |
| 30 hours | 50 GB | $p(10^{18})$ | $p(10^{19})$ |
| 100 hours | 150 GB | $p(10^{19})$ | $p(10^{20})$ |

# Sources of improvement

- Parallel computation (two threads)
- Faster exponential function
- Faster roots of unity
- Fewer terms in HRR series
- Faster FFT multiplication (Bill Hart)
- Slightly faster hardware

# Time breakdown



# In numbers

| $n$ | Mem | Pi | Exp | T1 | T2 | Wall | CPU |
|---|---|---|---|---|---|---|---|
| $10^{17}$ | 4.5 | 0.2 | 1.2 | 1.7 | 2.1 | 2.1 | 3.8 |
| $10^{18}$ | 13 | 0.8 | 4.5 | 6.5 | 5.8 | 6.5 | 12 |
| $10^{19}$ | 38 | 3 | 17 | 24 | 23 | 24 | 47 |
| $10^{20}$ | 128 | 12 | 66 | 94 | 111 | 111 | 205 |

Mem: GB, timings: hours

Images and history:
http://en.wikipedia.org/wiki/Hungarian_pengo